REACHING THE CORE THROUGH A RANDOM STABLE ALLOCATION MECHANISM

by

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ABSTRACT

The theory and evidence have proved that the stability of the matching produced by a centralized stable matching mechanism is crucial for the success of the mechanism. Suppose that, for democratic reasons, the organizer of a two-sided matching market is not allowed to use any of the optimal stable matching mechanisms for each side of the market. This problem is treated here for the College admission model, where Colleges have fixed preferences given by priority rankings over students. The strategic behavior of the participants in the game induced by an arbitrary stable matching mechanism may lead to outcomes that are not stable under the true preferences. However, if the stable matching for the preferences selected is picked at random the stability of the equilibrium outcome is recovered. Precise answers are given to the strategic questions raised.

Key words: stable matching, Nash equilibrium, mechanism, stable matching rule.

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INTRODUCTION

The College Admissions model is the well-known discrete many-to-one matching model with responsive and strict preferences introduced by Gale and Shapley in 1962. Many labor markets of firms and workers at the entrance level and admission markets of students to Colleges are two-sided matching markets that can be modeled as a College Admission market. The algorithms of Gale and Shapley allowed that, in the real world, centralized stable matching mechanisms for such markets could be designed to produce the optimal stable matching for the Colleges or the optimal stable matching for the students. The natural question is how the participants should behave in these mechanisms:

Should they report their true preferences or is there a smart way in which some informed participants can influence the outcome in their own favor by changing appropriately their preference list?

The first theorem in this direction is the Non-manipulability Theorem due to Dubins and Freedman (1981). These authors prove that, *if a stable matching mechanism (or stable matching rule) produces the optimal stable matching for the students when these agents reveal their true preferences, then the mechanism is non-manipulable individually and collectively by the students*.[2] [3] Roth(1984-a) showed, through an example, that when Colleges may have a quota greater than one, the mechanism that produces the optimal stable matching for the Colleges is manipulable by the Colleges.

The Non-manipulability Theorem was the starting point for new investigations on the strategic behavior of the agents. Since then the main focus has been the symmetric one-to-one matching market (every agent has quota of one), where students are allocated to Colleges according to the College–optimal stable matching rule. In this context, three important results appeared in the literature, following the theorem of Dubins and Freedman. The first one is due to Roth (1984-b): *In the strategic game induced by the College–optimal stable matching rule, the Nash equilibrium outcomes are stable under the true preferences*[4]*;* the second result is due to Gale and Sotomayor (1985-b): *In the strategic*

[2] The original proof of this result is quite long (about 20 pages); two shorter proofs are presented in Gale and Sotomayor (1985-a).
[3] This result generalizes Roth's (1982) and is a corollary of a more general theorem due to Demange, Gale and Sotomayor (1987).
[4] This result was proved by Roth by making use of the deferred acceptance algorithm. A direct proof is given in Gale and Sotomayor (1985-b).

game induced by the College–optimal stable matching rule, every stable matching under the true preferences is a Nash equilibrium outcome. These two results together provide a core implementation result for the Marriage model, via the Nash equilibrium concept. The third result is from Gale and Sotomayor (1985-b): *If the College–optimal stable matching rule is used, there will be an incentive for some student to misrepresent her preferences whenever more than one stable matching exists.*

Nevertheless, the strategic behavior of the agents under an arbitrary stable matching rule, other than the optimal stable matching rules for each side of the market, has not been explored, even when every agent has a quota of one.

The purpose of the present paper is to continue the line of investigation started by Dubins and Freedman, by addressing the many-to-one case. Let S and C denote the set of students and Colleges, respectively; P_S and P_C are the profile of the true preference lists for S and C, respectively, and q_C is the array of quotas for the Colleges. It is allowed that a stable matching mechanism produces any stable matching.

Firstly, we observe that the argument used in the proof of the last result mentioned above (Gale and Sotomayor, 1985-b) requires neither that the matching rule produces the optimal stable matching for the Colleges, nor that Colleges have a quota of one. Such observation led us to Proposition 1, that is the converse of the Non-manipulability Theorem: *If a stable matching mechanism does not produce the optimal stable matching for the students when they tell the truth, then the given mechanism is manipulable by at least one student.*

Next, without looking for wider generalities, we focus on a special case of the College admission model. For each College, the ranking of the students is defined by some pre-fixed and known priority criterion, that could be, for example, the students' grades in some test.[5] We consider the following class of matching mechanisms: the students may replace their profile of true preference lists P_S by any Q; a stable matching rule f (not necessarily the optimal stable matching rule for a given side of the market) selects a stable

[5] This model was first considered in Sotomayor (1996-b) and was inspired in the admission market for Graduate Schools of Economics in Brazil. This kind of model has been referred in the literature as school choice problem. See Abdulkadiroglu and Sonmez (2003), Balinski and Sonmez (1999), Ehlers and Klaus (2006-a,b) Ergin (2002) and Kesten (2006-a,b), among others.

matching for the market $M(Q)=(S,C,Q,\{P_C,q_C\})$. Since the set of agents, the quotas and the preference lists of the Colleges are fixed we denote the outcome by $f(Q)$. In the context of this model, the Impossibility Theorem of Roth (1982) does not apply and the Non-manipulability Theorem plus Proposition 1 provide *the existence and uniqueness of a strategy-proof stable matching mechanism.*

The question that remains to be approached is then if it is possible to get an extension of the core implementation result of Roth(1984-b) and Gale and Sotomayor (1985-b) to our setup. A partial result is obtained here: *The strong core correspondence is implemented by the College-optimal stable matching rule via the Nash equilibrium concept. Indeed, if an outcome is the College-optimal stable matching for a Nash equilibrium profile of strategies under some stable matching rule (not necessarily the College–optimal stable matching rule), then it is stable under the true preferences.* Nevertheless, as we will show, *the stability of the Nash equilibrium outcomes fails to hold under an arbitrary stable matching rule.* Also, there may be strategy profiles that are Nash equilibria under some stable matching rule and are not so under other stable matching rules. This, perhaps, explains why the centralized matching markets that have succeed in the real world have only contemplated the optimal stable matching rules.

Further, we address a problem of practical order: Imagine that some market organizer faces the problem of designing a stable matching mechanism and, therefore, of specifying a stable matching rule. Recall that a stable matching rule is a systematic way of picking a particular stable matching for the market defined by the strategy profile selected. Suppose that, for democratic reasons, the market organizer is not allowed to use any of the optimal stable matching rules. In general, there might be more than one stable matching for a given strategy profile. Then, almost always, he will be required to choose a stable outcome for each possible profile of preferences. As the number of agents increases, such specification becomes impracticable. The existent algorithms find the whole set of stable matchings, but they do not specify a particular matching, unless it is one of the optimal stable matchings. (In the simple situation of three students and two Colleges with quota of one, each agent can submit five different lists of preferences, so the number of possible preference profiles for the students is 5^3. This means that the stable matching rule must specify an outcome for each of the 125 preference profiles!) It is reasonable, therefore, to expect that the market organizer will want to allocate the students to the Colleges according to a matching chosen at random, with equal probability, among the stable matchings for the profile of revealed preferences. We will call such a rule a "*random stable matching rule"*. What are the effects on the strategic possibilities for the students caused by the adoption of a random stable matching rule?

To have some insight on that, suppose we fix some stable matching rule f and choose some profile of Nash equilibrium strategies for the students, Q, for the game induced by f. Suppose further that Q is not a Nash equilibrium under the stable matching rule f^*, and student s can profit by deviating from her strategy $Q(s)$. What might happen if one advises the students to follow Q under a random stable matching rule? Of course, it might be that the game they will play is not that one induced by f. Indeed, the students are not sure about which game they will play, but, presumably, they have prior beliefs about which stable matching rule will be used. Thus, if student s believes that she will play the game induced by f^* with a high probability, then it is possible that s will not follow Q! In this case, *is there any advice we could give to all the students so that, once it was adopted, would still be good advice for all of them to follow, taking in account their prior beliefs about which stable matching rule will be employed?* In other words, *is there a profile of strategies such that, ex-post, it is regret-proof for all students?*

Under these considerations, some good advice for the students will be *a strategy profile which is a Nash equilibrium under every possible stable matching rule*: no student has some chance, under some stable matching rule, of being better off by changing her strategy, given the others do not change theirs. This notion is the discrete version of the **Nash equilibrium concept in the strong sense**, introduced by Demange and Gale (1985) for the one-to-one matching model, in which the utilities are continuous in the money variable. In this concept, it is supposed that a student changes her strategy whenever she has a chance to be better off.

At the first glance, one might think that the process of reaching Nash equilibria in the strong sense, when they existed, would be very complicated when a large number of stable matching rules existed. It could then be that the students were not able to identify profitable strategic manipulations, whenever they were not sure about their payoffs. However, this conclusion turns out to be false. Indeed, the class of students' strategy profiles that may be profitable to be employed is a quite simple one. It consists of those preference profiles Q that meet two properties: a) The market *M(Q),* defined by Q and the fixed preferences and quotas of the Colleges, has a singleton set of stable matchings. Then, if μ is the only stable matching for *M(Q)*, and the students select Q, they will be sure that they will get μ, under the random stable matching rule. b) Q is a profile of Nash equilibrium strategies under some stable matching rule. We prove that to selecting a strategy profile that satisfies a) and b) is equivalent to choosing a strategy profile that is a Nash equilibrium under every stable matching rule. In fact, we characterize the Nash equilibria in the strong sense as the Nash equilibria Q of stable matching rules such that *M(Q)* has a singleton strong core**:** *a given profile of Nash equilibrium strategies under some stable matching rule is a Nash equilibrium in the strong sense, if and only if it realizes the same matching* μ *under every stable matching rule.*

This result is mathematically unusual. It provides a way of concluding that a strategy profile Q is not a Nash equilibrium for every game, based only on the direct examination of the size of the core for *M(Q),* without necessity of knowing the true preferences of the students. Since the definition of a Nash equilibrium involves the true preferences of the players, this is surprising. In fact we do not know of any comparable result concerning the equilibria of a game.

An important feature of the Nash equilibria in the strong sense, that we prove here, is *that the Nash equilibrium outcomes in the strong sense are stable under the true preferences.* If we consider that the stable matching rules that are feasible to be used are the student optimal, the College optimal and the random stable matching rules, then the stability of the equilibrium outcomes is a much more fundamental feature than one might expect.

As for the existence of equilibria, we show that *every stable matching under the true preferences is the outcome of a Nash equilibrium in the strong sense,* so it is a Nash equilibrium under any stable matching rule. This result provides an existence theorem, not only for the game where the matchings are selected at random, but also for any game induced by some stable matching rule.

For the special case in which the students play a truncation of the true preferences[6], we show that *the only equilibrium outcome in the strong sense is the student optimal stable matching under the true preferences.*

The extension of the strong core implementation result of Roth (1984-b) and Gale and Sotomayor (1985-b) to the model we are treating here is then obtained: *The random stable matching rule implements the core correspondence via the concept of Nash equilibrium in the strong sense and the student-optimal stable matching is the equilibrium outcome most preferred by the students.*

Finally, we address the strategic behavior of coalitions. The appropriate concept of equilibrium is that of strong equilibrium: *a profile of strategies such that no coalition of players can all gain by a simultaneous deviation (while the players outside the coalition maintain their strategies.* We provide an existence result which exhibits a profile of strategies, Q^*, which is a strong Nash equilibrium in the strong sense under the random stable matching rule. Furthermore, the optimal stable matching for the students is the resulting outcome [7]. Therefore, there is a sense in which Q^* can be regarded as the best method of playing for the students. It is (a) *a strong equilibrium in the strong sense, so no student or set of students will be tempted to deviate from it under any stable matching rule, and (b) among all equilibrium strategies, it gives the students the highest possible payoff.*

This paper is organized as follows. Section 2 gives the cooperative framework. We describe the stable mechanisms and present the new results in section 3. Sub-section 3.1 treats the stable matching rules. Sub-section 3.2 is devoted to the random stable matching rule. Section 4 concludes the paper. The proofs are presented in Appendix.

2. THE COLLEGE ADMISSION MARKET

There are two finite and disjoint sets: S with m students and C with n Colleges. For each College c there is a positive integer q(c) called the *quota* of c, which represents the number of positions it has to offer.

[6] The list of preferences Q(s) is a truncation of the true preferences P(s) at some agent v, if Q(s) ranks the Colleges in the same order as P(s), but ranks as unacceptable all Colleges which are ranked below v.
[7] This result generalizes Theorem 4 of Gale and Sotomayor (1985-b).

Each student has preferences over the Colleges and each College has preferences over individual students. We will assume these preferences are complete, transitive and strict, and so they may be represented by ordered lists, with $P(c)=s_1, s_3, s_2, c, s_4,\ldots$ denoting that College c prefers s_1 rather than s_3, that it prefers s_3 rather than s_2, that it prefers either one of them rather than leaving a position unfilled, and that all other students are unacceptable, in the sense that it would be preferable to leave a position unfilled rather than filling it with, say, student s_4. Similarly, $P(s)=c_1, c_2, s, c_3,\ldots$ represents the preferences of student s. We write $s >_{P(c)} s'$ to indicate that College c prefers s to s' and $s \geq_{P(c)} s'$ to mean c likes s at least as well as s', under the list of preference $P(c)$. Similarly, we write $c >_{P(s)} c'$ and $c \geq_{P(s)} c'$. College c is *acceptable* to student s under P if $c >_{P(s)} s$, and student s is *acceptable* to College c if $s >_{P(c)} c$. We will usually abbreviate a preference list to include just the acceptable alternatives.

We will assume that the preferences of a College over groups of students are *responsive* to its preferences over individual students. That is, if A is a set of students such that $|A| < q(c)$, and s and s' are not in A, then College c prefers $A\cup\{s\}$ to $A\cup\{s'\}$ if and only if $s >_{P(c)} s'$. If $|A|\leq q(c)$ and $|B|\leq q(c)$, we will write $A >_{P(c)} B$ to denote that College c prefers A to B and $A \geq_{P(c)} B$ to denote that College c likes A at least as well as B. For our purposes, we need not know the preferences of the Colleges over groups of students. Then, the preferences of students and Colleges are given by $P=\{P_S,P_C\}$, where P_S is the profile of preference lists of the students and P_C is the profile of preference lists of the Colleges over individual students. Denoting by q_C the array of quotas of the Colleges, the market is then given by $M=(S,C,P_S,\{P_C,q_C\})$.

The rules of the market allow that any student and College may mutually agree that the College will admit the student; any College may choose to keep any of its positions unfilled, and any student may remain unmatched if she wishes. A matching is an allocation of students to Colleges that respects the quotas of the agents. A student who is not matched to any College will be matched to herself and will be called unmatched; a College that has some number of unfilled positions will be matched to itself in each of these positions. If μ is a matching, $\mu(s)=c$ denotes that student s is admitted by College c at the matching μ; $\mu(c)=\{s_2, s_4, c, c\}$, for example, denotes that College c, with quota

$q(c)$=4, admits students s_2 and s_4 and has two unfilled positions . Given any two matchings μ and μ' and any agent $y \in S \cup C$ we write $\mu >_{P(y)} \mu'$ to mean $\mu(y) >_{P(y)} \mu'(y)$ and $\mu \geq_{P(y)} \mu'$ to mean $\mu(y) \geq_{P(y)} \mu'(y)$. For the market M we define:

Definition 1. *A matching μ is* ***feasible*** *for M if for every pair (s,c), such that $\mu(s)=c$, s and c are mutually acceptable under P.*[8]

The adequate concept of stability for the matching models is that of setwise-stability (see Sotomayor, 1999). For the College admission model, setwise-stability is equivalent to pairwise-stability.

Definition 2. *A matching μ is* ***pairwise-stable*** *(stable, for short) for the market M if it is feasible for M and there is no student s and College c, not matched to one another at μ, such that a) $c >_{P(s)} \mu(s)$ and b) $s >_{P(c)} \sigma$, for some $\sigma \in \mu(c)$.*

A student s and a College c, as in Definition 2, are said *to cause an instability in the matching μ.* For an abuse of language we will use, sometimes, the terminology " *(s,c)* blocks *μ*", which is more adequate for the cases in which this coalition can improve upon. The existence of stable matchings was first proved in Gale and Shapley (1962) by means of a well-known deferred-acceptance algorithm. A non-constructive proof is given in Sotomayor (1996-a). A more general proof of the existence of pairwise-stable matchings is given in Sotomayor (1999) for the discrete many-to-many matching model with substitutable preferences and non-necessarily strict preferences. It can be easily seen that the set of stable matchings coincides with the strong core.

Definition 3. *For a given market M, a stable matching μ_S is called* ***the optimal stable matching for the students*** *(or* ***S-optimal stable matching****) if $\mu_S(s) \geq_{P(s)} \mu(s)$ for every*

[8] For stability purposes, Definition 1 could require individual rationality instead of mutual acceptance. The reason for our choice is that under stable matchings, if two players form a partnership then they are mutually acceptable. Clearly, feasibility implies individual rationality, but the converse is not true.

student s *and for every stable matching* μ*. The* ***optimal stable matching for the Colleges*** *(or the* ***C-optimal stable matching)*** μ_C ***,*** *is analogously defined.*

The following terminology will be used. We say that the list of preference $Q(s)$ is *a truncation of* $P(s)$ *at agent* v (v might be s), if $Q(s)$ ranks the Colleges in the same order as $P(s)$, but ranks as unacceptable all Colleges which are ranked below v. We say that $Q(s)$ *is a truncation of* $P(s)$ if it is a truncation of $P(s)$ at some agent v.

Given the set of students S and the set of Colleges C, we will use the following notation:

> P_C is the profile of true preferences for the Colleges; P_S is the profile of true preferences for the students.
>
> $\Re(s)$ is the set of all possible ordered preference lists $Q(s)$ of student s.
>
> q_C is the array of quotas for the Colleges.

If Q is a profile of preferences for the students:

> $M(Q)=(S,C,Q,\{P_C,q_C\})$ is the cooperative market game associated to Q. We set $M\equiv M(P_S)$.
>
> $\mu_S(Q)$ is the S-optimal stable matching for $M(Q)$. We set $\mu_S\equiv\mu_S(P_S)$.
>
> $\mu_C(Q)$ is the C-optimal stable matching for $M(Q)$. We set $\mu_C\equiv\mu_C(P_S)$.
>
> $E(Q)$ is the set of stable matchings for $M(Q)$.
>
> Q_{-T} indicates the restriction of Q to S-T, where $T\subseteq S$. If $T=\{s\}$ we will simply write Q_{-s} instead of $Q_{-\{s\}}$.

Table 1.

3. THE STABLE MATCHING RULES

In this section we consider the following matching mechanisms for the model we are treating here. Given the market $(S,C,P_S,\{P_C,q_C\})$, each student $s\in S$ may replace her true preference, $P(s)$, by any list of preference $Q(s)\in\Re(s)$. Let Q denote the profile of lists of preference $Q(s)$, one list for each student $s\in S$. Once Q is selected, S, C, Q and $\{P_C,q_C\}$ are used as "input" for some algorithm that yields a stable matching for the market $(S,C,Q,\{P_C,q_C\})$, as a final "output". Since S, C and $\{P_C,q_C\}$ are fixed we denote such market by $M(Q)$. This procedure is described by a function f that we will be called *stable* *matching rule*. For each profile of preferences Q, $f(Q)$ is the stable matching for $M(Q)$, selected by f.

Two special stable matching rules are the student-optimal stable matching rule and the College-optimal stable matching rule. Under the first one (respectively second one) the agents are assigned in accordance with the S-optimal (respectively C-optimal) stable matching for $M(Q)$.

3.1. STRATEGIC POSSIBILITIES FOR THE STUDENTS UNDER A STABLE MATCHING RULE

The adoption of a stable matching rule f (or any mechanism with the stable matching rule f) for a given market $(S,C,P_S,\{P_C,q_C\})$, induces a strategic game where: P_C and q_C are fixed; the set of players is the set of students S; a strategy of student s is any list of preferences $Q(s)$ in $\Re(s)$; the outcome function is given by f and the true preferences of the players are given by P_S. We say that Q *realizes* $f(Q)$. We denote this game by $\Gamma(f)$ and we refer to it as the *admission game* induced by f. Sometimes we will refer to a stable matching for $M(Q)$ as a *stable matching under* Q.

A stable matching rule f will be called *strategy-proof* (or individually non-manipulable) if it makes it a dominant strategy for each student s to state her true preference $P(s)$ in the strategic game it induces. We say that the stable matching rule f is *manipulable by a coalition* $T\subseteq S$, if there is a preference profile Q for the students, with $Q_{-T}=P_{-T}$, such that $\mu'(s) >_{P(s)} \mu(s)$ for all $s\in T$, where $\mu\equiv f(P_S)$ and $\mu'\equiv f(Q)$.

The Non-manipulability Theorem of Dubins and Freedman (1981) (see Appendix, Proposition 4*) implies that every stable matching rule f, with $f(P_S)=\mu_S$, is not collectively manipulable. Therefore, the mechanisms that remain to be investigated are those for which $f(P_S)\neq\mu_S$. Our first result in this direction is:

Proposition 1. *Let f be a stable matching rule with $f(P_S)=\mu$. If $\mu\neq \mu_S$ then f is manipulable by every student s such that $\mu_S(s)\neq\mu(s)$.*

Thus, if f is not collectively manipulable, it is not individually manipulable, so $f(P_S) =\mu_S$. Proposition 1 plus the Non-manipulability Theorem characterize the stable matching mechanisms that are strategy-proof. Due to the importance of this result we will state it formally:

Theorem 1. *Let f be a stable matching rule. Then, f is strategy-proof if and only if $f(P_S)=\mu_S$.*

Observe that in Theorem 1 it is not required that f is the student-optimal stable matching rule. Let us illustrate Theorem 1 with the following example:

Example 1. Let $S=\{s_1,s_2,s_3,s_4\}$, $C=\{c_1,c_2,c_3\}$ and $q(c_1)=2$, $q(c_2)=q(c_3)=1$. The profile of preferences P_S and P_C are given by:

$P(s_1)=c_3, c_2, c_1$	$P(c_1)=s_1, s_2, s_4, s_3$
$P(s_2)=c_1$	$P(c_2)=s_3, s_4, s_1$
$P(s_3)=c_1, c_3, c_2$	$P(c_3)=s_4, s_3, s_1$
$P(s_4)=c_2, c_1, c_3$	

The C-optimal stable matching is given by: $\mu_C(s_1)=c_1$, $\mu_C(s_2)=c_1$, $\mu_C(s_3)=c_2$ and $\mu_C(s_4)=c_3$; the S-optimal stable matching is given by: $\mu_S(s_1)=c_3$, $\mu_S(s_2)=c_1$, $\mu_S(s_3)=c_1$ and $\mu_S(s_4)=c_2$. Consider the stable matching rule: $f(P_S)=\mu_C$ and $f(Q)$ is arbitrary if $Q\neq P_S$. Observe that $f(P_S)\neq\mu_S$. Now, let s_4 misrepresent her preference and state $Q(s_4)=c_1, c_2$. Set $Q\equiv(P(s_1), P(s_2), P(s_3), Q(s_4))$. It is a matter of verification that s_4 gets her second choice, c_1, if $f(Q)=\mu_S(Q)$. Hence, by Proposition 1* of Appendix, s_4 will be matched to c_1 or c_2 under any stable matching rule. In any case, s_4 will profit from stating $Q(s_4)$ instead of $P(s_4)$. ■

Remark 1. We must point out that Proposition 1 holds when preferences of the Colleges are not fixed. Then, it generalizes Theorem 1 of Gale and Sotomayor (1985-b), where the College optimal stable matching rule is used for a one-to-one matching model. ■

As indicated by Proposition 1 for the case in which there is more than one stable matching for M, the stable matching rules f, with $f(P_S)\neq\mu_S$, are always manipulable by at least one student. It is therefore natural to try to predict the result of such manipulation, assuming *the students have unlimited information and possibilities for communication with each other.* The adequate concept of equilibrium is provided by that of Nash equilibrium.

Definition 4. *Let* $\Gamma(f)$ *be an admission game. The profile of strategies* Q *is a* ***Nash equilibrium*** *of* $\Gamma(f)$ *if for every student* s, $f(Q) \geq_{P(s)} f(Q'(s), Q_{-s})$ *for every strategy* $Q'(s) \in \Re(s)$.

That is, a set of strategies, one for each student, forms a Nash equilibrium if no student, by changing her strategy, can get a better payoff assuming the other students do not change their strategies.

If Q is a Nash equilibrium of $\Gamma(f)$, we say that $f(Q)$ **is the Nash equilibrium outcome corresponding to** Q**.**

The existence of Nash equilibria under any stable matching rule is provided by Theorem 2 below:

Theorem 2. *Let* μ *be a stable matching for* M*. Suppose each student* $s \in \mu(C)$ *chooses the strategy* $Q(s)$ *of listing only* $\mu(s)$ *as the only acceptable College;* $Q(s)=s$ *in case* s *is unmatched at* μ*. The profile of strategies* Q *is a Nash equilibrium of every admission game and* μ *is the equilibrium outcome corresponding to* Q*.*

For the special case in which every College has a quota of one and the College-optimal stable matching rule is used, Roth (1984-b) and Gale and Sotomayor (1985-b) proved that if Q is a Nash equilibrium then $\mu_C(Q)$ is stable under the true preferences. Proposition 2 provides a more general result. It requires neither that the Colleges have a quota of one and nor that the stable matching rule always selects the *C*-optimal stable matching. That is, the stable matching rule f may be such that $f(Q)=\mu_C(Q)$ and $f(Q')$ is arbitrary if $Q' \neq Q$. Formally,

Proposition 2. *Let* Q *be a preference profile. Let* f *be a stable matching rule such that* $f(Q)=\mu_C(Q)$**.** *If* Q *is a Nash equilibrium for* $\Gamma(f)$, *then* $\mu_C(Q)$ *is stable for the market* M.

In particular, if Q is a Nash equilibrium of the game induced by the College-optimal stable matching rule, then $\mu_C(Q)$ is stable under the true preferences. This observation plus Theorem 2 leads to:

Theorem 3. *The College-optimal stable matching rule implements the strong core correspondence via the Nash equilibrium concept.*

When the stable matching rule is arbitrary, we may have Nash equilibrium outcomes, which are unstable under the true preferences. See the example below.

Example 2. (μ^1 *is a Nash equilibrium outcome but it is unstable under the true preferences*). Let $S=\{s_1,s_2,s_3\}$, $C=\{c_1,c_2,c_3\}$ and $q(c_j)=1$, for all $j=1,2,3$. Then, let P_S, P_C and Q be given by:

$P(s_1)= c_1, c_2, c_3$	$P(c_1)= s_2, s_1, s_3$	$Q(s_1)= c_2, c_3$
$P(s_2)= c_3, c_2, c_1$	$P(c_2)= s_3, s_1, s_2$	$Q(s_2)=c_3, c_1$
$P(s_3)= c_1, c_2, c_3$	$P(c_3)= s_1, s_3, s_2$	$Q(s_3)= c_1, c_3, c_2$

The stable matchings under Q are μ^1, μ^2 and μ^3, where $\mu^1(s_1)=c_2$, $\mu^1(s_2)=c_3$, $\mu^1(s_3)=c_1$; $\mu^2(s_1)=c_2$, $\mu^2(s_2)=c_1$, $\mu^2(s_3)=c_3$ and $\mu^3(s_1)=c_3$, $\mu^3(s_2)=c_1$, $\mu^3(s_3)=c_2$.

We claim that Q is a Nash equilibrium of every game in which Q realizes μ^1. In fact, since μ^1 matches s_2 and s_3 to their first true choice, we only need to check the deviations of s_1. Then suppose by contradiction that there is some stable matching rule f, with $f(Q)=\mu^1$, and some profile of preferences $Q'=(Q'(s_1),Q(s_2),Q(s_3))$ such that s_1 prefers $f(Q')$ to μ^1. Set $\mu'\equiv f(Q')$ Then $\mu'(s_1)=c_1$. Now, observe that, if $\mu'(s_2)=c_3$ then (s_3,c_3) causes an instability in μ' under Q' If s_2 is not matched to c_3, s_2 is unmatched. But then (s_2,c_1) causes an instability in μ' under Q'. Thus, in any case we contradict the stability of μ' under Q'. Therefore, $\boldsymbol{Q}$ **is a Nash equilibrium of every game in which**

***Q* realizes μ^1**. It is a matter of verification that (s_1,c_1) causes an instability in μ^1 under the true preferences, so μ^1 is unstable under P_S. ■

The following result is a corollary of Proposition 2. It guarantees the stability of the equilibrium outcome under the true preferences.

Corollary 1. Let *f be any stable matching rule. Let Q be a Nash equilibrium of the game Γ(f). If* $|E(Q)|=1$ *then the corresponding Nash equilibrium outcome is stable under* P_S.

Observe that in Example 2, μ^1 is the most preferred outcome of $E(Q)$ by the students. This fact is not accidental, as shown by Proposition 3.

Proposition 3. *Let Q be a Nash equilibrium for Γ(f). Let f(Q)=μ. Then* $\mu(s) \geq_{P(s)} \mu'(s)$, $\forall s \in S$ *and* $\mu' \in E(Q)$.

If cooperation among students is possible, these agents might cooperate to influence the outcome towards their interest. In such cases the behavior of the coalitions should be considered. The appropriate concept of equilibrium is that of strong Nash equilibrium: *a profile of strategies such that no coalition of players can all gain by a simultaneous deviation (while the players outside the coalition maintain their strategies).* Formally,

Definition 5. *Let Γ(f) be an admission game. The profile of strategies Q is a* ***strong Nash equilibrium*** *of Γ(f) if there is no coalition S' of students and a profile of strategies Q' such that* $f(Q'(s),Q_{-s}) >_{P(s)} f(Q)$ *for every* $s \in S'$.

Theorem 4 shows the existence of strong Nash equilibrium.

Theorem 4. *For each student s, let Q*(s) be a truncation of P(s) at* $\mu_S(s)$. *Then Q* is a strong Nash equilibrium under every stable matching rule. Furthermore* μ_S *is the resulting matching.*

3.2 THE STRATEGIC POSSIBILITIES FOR THE STUDENTS UNDER A RANDOM STABLE MATCHING RULE

This sub-section address the problem of choosing a stable matching rule, other than the optimal stable matching rules for each side of the market. As discussed in section 1, it may be impracticable to specify an arbitrary stable matching for every possible preference profile. A way of solving this problem is to use a *random stable matching rule*, that is, a matching rule that picks a matching at random, with equal probability, among the stable matchings for the selected profile of preferences. The question is: How is the equilibrium concept affected by the change of the rules of the game? As argued in section 1, the adequate concept of equilibrium is the discrete version of the well-known concept of Nash equilibrium in the strong sense, defined by Demange and Gale (1985) for the continuous one-to-one matching model. **A strategy profile is a Nash equilibrium in the strong sense if it is a Nash equilibrium for every possible admission game.** The definition of strong Nash equilibrium in the strong sense is straightforward. Theorem 5 characterizes the Nash equilibrium in the strong sense.

Theorem 5. *Let Q be a Nash equilibrium of some admission game. Then Q is a Nash equilibrium in the strong sense if and only if $|E(Q)|=1$.*

Thus, it is enough to take some stable matching rule and then find some Nash equilibrium with singleton strong core for the corresponding game. This will be a Nash equilibrium in the strong sense.

The existence of Nash equilibrium profile of preferences that have singleton strong core is provided by Theorem 2. Thus, we obtain the existence of strategy profiles that are Nash equilibria for every admission game, independent of the stable matching rule. Of course, not all Nash equilibria have singleton strong core. In Example 2, Q is a Nash equilibrium when it realizes μ^1 and $M(Q)$ has three stable matchings. In this example, Q is not a Nash equilibrium when it realizes μ^2 (s_2 can profit by deviating and selecting $Q'(s_2)=c_3$).

An immediate application of Theorem 5 and Corollary 1 leads to:

Theorem 6. *Let Q be a Nash equilibrium in the strong sense. Then the corresponding Nash equilibrium outcome is stable under the true preferences.*

Proposition 4 implies that μ_S is the only equilibrium outcome in the strong sense obtained via truncation of the true preferences:

Proposition 4. *For each student s, let $Q(s)$ be a truncation of $P(s)$. If the profile of strategies Q is a Nash equilibrium in the strong sense then μ_S is the equilibrium outcome.*

Proposition 5 characterizes the truncations of P_S that are strong Nash equilibrium in the strong sense. Formally,

Proposition 5. *For each student s, let $Q(s)$ be a truncation of $P(s)$. Then Q is a strong Nash equilibrium in the strong sense if and only if $E(Q)=\{\mu_S\}$.*

Theorem 4 provides the existence of strong Nash Equilibrium in the strong sense. Theorem 2 implies that the students can force any matching μ, which is stable under the true preferences, via Nash equilibrium in the strong sense. In particular they can force the student-optimal stable matching under the true preferences. By Theorem 4, the students also can force μ_S via strong Nash equilibrium in the strong sense. This way, the strategy described in Theorem 4 could be regarded as the best method of playing for the students: It is (a) *a strong equilibrium in the strong sense, so no student or set of students will be tempted to deviate from it under any stable matching rule, and (b) among all equilibrium strategies, it gives the students the highest possible payoff.*

The desired result is trivially obtained via Theorems 2 and 6:

Theorem 7. *The random stable matching rule implements the strong core correspondence via the Nash equilibrium in the strong sense concept.*

4. CONCLUDING REMARKS

This paper analyzes the class of games, which are induced by stable matching mechanisms, for a special case of the College admissions model. Colleges have any quota, but fixed preferences, given by priority rankings over students. First, we extended the core implementation result of Roth (1984-b) and Gale and Sotomayor (1985-b) to this many-to-one matching model and proved that it fails to hold under an arbitrary stable matching mechanism. Next, we proposed a mechanism that picks any stable matching for the profile of revealed preferences with equal probability. For the induced "game", we defined a new concept of strategic equilibrium. We characterized such equilibria as those preference profiles which have a singleton strong core and are Nash equilibria of some stable matching rule. Then we used this equilibrium concept to get a general core implementation result.

The model treated here was first considered in Sotomayor (1996-b), motivated by the admission market for Graduate Schools of Economics in Brazil. In this market, the students are submitted to five tests: Mathematics, Statistics, Microeconomics, Macroeconomics and a test about the Brazilian Economy. The schools attribute weights to each one of the five subjects. The quotas and weights of the Colleges and the students' scores in the tests are common knowledge. The students are evaluated by the Colleges in accordance to their weighted average score in the tests. These evaluations are used by the Colleges to determine a rank of the students. In 1997, a centralized matching mechanism for this market used the Gale and Shapley's algorithm that produces the student-optimal stable matching. That paper shows the converse of the Theorem of Dubins and Freedman, the existence of Nash equilibria and strong Nash equilibria for a given stable matching rule. For the College optimal stable matching rule, it proves that every Nash equilibrium outcome is stable under the true preferences and characterizes all the dominated strategies for a given student. This paper was published in Portuguese. Then, for the sake of completeness, some of its results were presented here.

The results of the present paper can be used to analyze the strategic behavior of both kinds of agents, students and Colleges, in a two-stage mechanism for the College Admission market of Gale and Shapley. At the first stage, simultaneously, Colleges select their strategy preference profile and quotas. At the second stage, knowing the profile of strategies selected by the Colleges, the students choose, simultaneously, their strategy preference profile. Then, the College optimal stable matching rule is applied. The reader can identify the sub game played by the students in the second stage with the game treated in this paper. Of course, the adequate concept of equilibrium is now that of sub game perfect Nash equilibrium. According to our results, in each sub game, the students can force, by Nash equilibria, the student optimal stable matching under their true preferences and the profile of preferences stated by the Colleges in the first stage. The analysis of the strategic possibilities for the Colleges in this mechanism is a problem which we intend to investigate in the future.

The implementability of the core correspondence through stable mechanisms, which are closely related to Gale and Shapley's algorithms, was also investigated by several authors. We can cite, among others, Gale and Sotomayor (1985-b), Roth (1984-b), Alcalde (1996), Alcalde and Romero-Medina (1996), Alcalde, Perez-Castrillo and Romero-Medina (1997) and Sotomayor (2002). The common feature of these mechanisms is that the equilibrium outcomes are stable under the true preferences. Sotomayor (2003) proved that this property also holds for a mechanism, which is not designed for producing stable outcomes. For the case where colleges have fixed preferences, Balinski and Sonmez (1999) noted that since the welfare of students matters, the student-optimal stable mechanism Pareto dominates any other stable mechanism. Ergin (2002) showed that the student-optimal stable mechanism is Pareto efficient if and only if the preference structure of the colleges is "acyclic".

5. APPENDIX

This section is devoted to the proofs of the propositions presented in section 3. As assumed in the text, the preferences over individuals are strict. The following results, whose proofs can be found in Roth and Sotomayor (1990), will be needed:

Proposition 1*. (Gale and Sotomayor (1985-a)) *The set of students admitted and positions filled in a college admissions market is the same at every stable matching.*

Proposition 2*. (Gale and Sotomayor (1985-a) *Suppose* $S' \subseteq S$ *and let* μ_S, μ_C, μ'_S, and μ'_C *be the optimal stable matchings corresponding to* (S,C,P,q_C) *and* (S',C,P',q_C), *respectively, where* P'_S *agrees with* P_S *on* S'. *Then, for every College* c,

$$\mu_S(c) \gtrsim_{P(c)} \mu'_S(c) \text{ and } \mu_C(c) \gtrsim_{P(c)} \mu'_C(c).$$

Remark 2. *It follows from the proof of Proposition 2* that for all* $s \in \mu_S(c)$ *(respectively* $s \in \mu_C(c)$*), there is some* $s' \in \mu'_S(c)$ *(respectively* $s' \in \mu'_C(c)$*) such that* $s \gtrsim_{P(c)} s'$. ■

Proposition 3*. (Roth and Sotomayor (1989)) *Let* μ *and* μ' *be stable matchings for* M. *Let* $c \in C$. *Then, a)* c *is indifferent between* $\mu(c)$ *and* $\mu'(c)$ *if and only if* $\mu(c) = \mu'(c)$; *b) If* $\mu(c) >_{P(c)} \mu'(c)$ *then* $s >_{P(c)} s'$ *for all* $s \in \mu(c)$ *and* $s' \in \mu'(c) - \mu(c)$.

Remark 3. ***If*** Q ***is a Nash equilibrium of some admission game in which*** Q ***realizes*** μ***, then*** μ ***is individually rational for the students under the true preferences, and so it is feasible.*** *In fact, if* $s >_{P(s)} \mu(s)$ *for some student* s, *then* s *can be better off by choosing* $Q'(s) = \{s\}$, *because then she will be unmatched at any stable matching for* $M(Q'(s),Q_{-s})$. *But this contradicts the assumption that* Q *is a Nash equilibrium of the given game.* The feasibility of μ then follows from the fact that P_C is the preference profile of the Colleges and μ is stable under $\{Q,P_C\}$. ■

Proposition 4*. (Non-manipulability Theorem - Dubins and Freedman (1981)) *Let* Q *differ from* P_S *in that some coalition* $S' \subseteq S$ *misstate its preferences. Then there is no matching* μ, *stable for* $(S,C,Q,\{P_C, q_C\})$ *which is preferred to* μ_S *by all members of* S'.

The proofs of Proposition 1 and Theorem 2 are, respectively, adaptations of the proofs of Theorems 1 and 2 of Gale and Sotomayor (1985-b). The proof of Proposition 2 makes strong use of Proposition 3*, which has no parallel in the one-to-one matching model.

The idea of the proof of Proposition 1 is to show that truncating the true preferences at $\mu_S(s)$, when $\mu_S(s) \neq \mu(s)$, yields a better assignment than $\mu(s)$.

Proof of Proposition 1. Let s be such that $\mu(s) \neq \mu_S(s)$. Then,

$$\mu_S(s) >_{P(s)} \mu(s) >_{P(s)} s, \qquad (1)$$

where the last inequality is due to Proposition 1* (if $\mu(s)=s$, then $\mu_S(s)=s=\mu(s)$, contradiction). Let $Q(s)$ be the truncation of $P(s)$ at $\mu_S(s)$. Set $\mu' \equiv f(\{Q(s),P_{-s}\})$. We claim that $\mu'(s) \geq_{P(s)} \mu_S(s)$. In fact, first observe that μ_S is stable for $M'=(S,C,\{Q(s),P_{-s}\},\{P_C, q_C\})$. Thus, since s is matched at μ_S, and μ' is stable for M', Proposition 1* implies that s will be matched at μ'. Hence, $\mu'(s) \geq_{P(s)} \mu_S(s)$, by the construction of $Q(s)$. Using (1) we get that $\mu'(s) >_{P(s)} \mu(s)$. Hence s can improve her payoff by misrepresenting her preferences. ■

Proof of Theorem 2. It is clear that μ is stable for the market $M(Q)=(S,C,Q,\{P_C,q_C\})$. Furthermore, μ is the only stable matching for $M(Q)$, for any other matching would leave some student s in $\mu(C)$ unmatched, which is not possible by Proposition 1*. Hence

μ is the matching produced by any stable matching rule when the students select *Q*. (1)

To see that Q is a Nash equilibrium for every admission game, suppose by way of contradiction that some student s changes her strategy from $Q(s)$ to $Q'(s)$, yielding a new profile of strategies $Q'=\{Q'(s),Q_{-s}\}$. Suppose further that there are f, $\mu' \in E(Q')$ and c, with $f(Q')=\mu'$ and $\mu'(s)=c$, such that $c >_{P(s)} \mu(s)$. Then, College c must have filled its quota under μ and prefers any student assigned to it at μ rather than student s, for if not (s,c) would cause an instability for μ in M. But then, some student s', who would have been assigned to c at μ, would not have been assigned to c at μ'. But if this were so, since c was the only acceptable mate to s', it would follow that s' would be unmatched at μ', and so (s',c) would have caused an instability in μ' in $(S,C,Q',\{P_C,q_C\})$, which is a contradiction. Hence, Q is a Nash equilibrium under any stable matching rule, so it is a Nash equilibrium in every admission game. Now use (1) to get that μ is the Nash equilibrium outcome corresponding to Q. ■

Proof of Proposition 2. Let f be any stable matching rule such that Q is a Nash equilibrium of $\Gamma(f)$ and $f(Q)=\mu_C(Q)$. For simplicity of notation set $\mu_C(Q)\equiv\mu$. Suppose, by way of contradiction, that μ is not stable under P_S. Since μ is feasible under the true preferences by Remark 3, then there exists a pair, (s,c), that causes an instability in μ and so

$$c >_{P(s)} \mu(s) \text{ and } s >_{P(c)} \sigma, \text{ for some } \sigma\in\mu(c) \ (\sigma \text{ may be } c). \qquad (1)$$

We will show that Q is not a Nash equilibrium of $\Gamma(f)$, which is a contradiction. To see this, let s deviate from $Q(s)$ by listing only c on her list of acceptable Colleges. Set $\mu'\equiv f(Q')$, where Q' differs from Q only in the new list of s. If s does not get c, then she is unmatched at μ'. By Proposition 1*, s is also unmatched at $\mu'_C\equiv \mu_C(Q')$.

On the other hand, the stability of μ'_C in $M(Q')$ and the fact that $\mu'_C(s)=s$ imply that College c fills its quota under μ'_C and

$$s' >_{P(c)} s, \quad \forall s'\in \mu'_C(c). \qquad (2)$$

By (1), (2) and the transitivity of the preferences,

$$s' >_{P(c)} \sigma, \quad \forall s'\in \mu'_C(c). \qquad (3)$$

Now, observe that the restriction of matching μ'_C to $S'\equiv S-\{s\}$ is stable for the market $(S',C, Q_{-s},\{P_C,q_C\})$. Let μ_C^* denote the C-optimal stable matching for $(S',C, Q_{-s},\{P_C, q_C\})$. Then, $\mu_C^*(c) \geq_{P(c)} \mu'_C(c)$, by the optimality of μ_C^*. Since c fills its quota at μ'_C, Proposition 1* implies that

$$\textbf{\textit{c} fills its quota at } \mu_C^*. \qquad (4)$$

By Proposition 3*, either $\mu_C^*(c)= \mu'_C(c)$ or $\mu_C^*(c) >_{P(c)} \mu'_C(c)$. In the last case c prefers any student in $\mu_C^*(c)$ rather than any student in $\mu'_C(c)- \mu_C^*(c)$. In any case, using (3),

$$\textbf{\textit{c} prefers any student in } \mu_C^*(c) \textbf{ rather than } \sigma. \qquad (5)$$

From Proposition 2* we have that $\mu(c) \geq_{P(c)} \mu_C^*(c)$, and from Remark 2 it follows that College c weakly prefers σ to some of its mates in $\mu_C^*(c)$. By (4), all c's mates at μ_C^* are students, so c weakly prefers σ to some student in $\mu_C^*(c)$, which contradicts (5). Hence $\mu'(s)=c$, so Q is not a Nash equilibrium of the game $\Gamma(f)$, as we wanted to show. ■

Proof of Corollary 1. Let μ be the only stable matching in $E(Q)$. Then μ is the C-optimal stable matching under Q, so Proposition 2 applies. Hence μ is stable under P_S, and the proof is complete. ■

Proof of Proposition 3. Suppose by way of contradiction that $\mu'(s) >_{P(s)} \mu(s)$ for some $s \in S$ and $\mu' \in E(Q)$. The individual rationality of μ implies that $\mu'(s) \equiv c$ for some $c \in C$. Let $Q'(s) = c, s$ and $Q'_{-s} = Q_{-s}$. It is clear that μ' is stable under Q'. By Proposition 1*, s is matched at every stable matching under Q', so s gets c under f when the students play Q', so s gets better by deviating from Q, and so Q cannot be a Nash equilibrium of the game $\Gamma(f)$, which is a contradiction. ■

Proof of Theorem 4. We claim that μ_S is the only stable matching for $M(Q^*)=(S,C,Q^*,\{P_C,q_C\})$, for clearly μ_S is stable for $M(Q^*)$ and any other stable matching μ' must have $\mu'(s) \neq \mu_S(s)$ for some s; hence Proposition 1* implies that s is matched at μ'. Then $\mu'(s) >_{P(s)} \mu_S(s)$ by construction of Q^*. Since μ_S is the student optimal stable matching under P, we have that μ' is unstable under the true preferences, so it is blocked by some pair (s',c). By construction of Q^*, (s',c) also causes an instability in μ' under Q^*, contradicting the stability of μ' for $M(Q^*)$.

Now, since μ_S is the only stable matching under $M(Q^*)$, it is the matching yielded by any stable matching rule when Q^* is played. Therefore the conclusions of Proposition 4* apply and Q^* is a strong Nash equilibrium under every matching rule. The other assertion follows from the fact that $E(Q^*)=\{\mu_S\}$. ■

Proof of Theorem 5. Suppose Q is a Nash equilibrium of every admission game. If μ and μ' are stable matchings for Q, then Q is a Nash equilibrium of the games $\Gamma(f)$ and $\Gamma(f')$, where $f(Q)=\mu$ and $f'(Q)=\mu'$. In this case, Proposition 3 plus the strictness of the preferences would imply that $\mu=\mu'$. Hence, $|E(Q)|=1$.

In the other direction, suppose that Q is a Nash equilibrium of $\Gamma(f)$, with $f(Q)\equiv\mu$ and $E(Q)=\{\mu\}$. Now, suppose Q is not a Nash equilibrium of the admission $\Gamma(f')$, for some stable matching rule f'. Since $E(Q)=\{\mu\}$, we have that $f'(Q)=\mu$. Then there is some student s and $Q'=(Q'(s),Q_{-s})$, such that $f'(Q') >_{P(s)} \mu$. Set $\mu' \equiv f'(Q')$. By Remark 3, μ is individually rational for M, so $\mu'(s) >_{P(s)} \mu(s) \geq_{P(s)} s$, which implies that s must be matched to some c at μ'. Hence

$$c >_{P(s)} \mu(s) \qquad (1)$$

Let $Q''(s)=c$ and $Q''_{-s}=Q_{-s}$. It is clear that μ' is stable under Q''. Then, by Proposition 1*, s is matched at every stable matching under Q'', so s gets c under f when the students play Q''. But then (1) implies that the deviation $Q''(s)$ is profitable to s in the game $\Gamma(f)$, which contradicts the assumption that Q is a Nash equilibrium for $\Gamma(f)$. Hence Q is a Nash equilibrium in every game as we wanted to show. ■

Proof of Proposition 4. If Q is a Nash equilibrium in the strong sense then $|E(Q)|=1$ by Theorem 5. Then set $E(Q) \equiv \{\mu\}$. Theorem 6 implies that μ is stable for M, so the optimality of μ_S implies $\mu_S(s) \geq_{P(s)} \mu(s) \quad \forall s \in S$, so $\mu_S(s) \geq_{Q(s)} \mu(s) \quad \forall s \in S$, by construction of Q, and so μ_S is stable for $M(Q)$. But then, $\mu=\mu_S$ and we have completed the proof. ■

Proof of Proposition 5. The proof that the condition is necessary follows from Proposition 4. To see that the condition is sufficient, suppose $E(Q)=\{\mu_S\}$. Then μ_S is the student optimal stable matching for $M(Q)$. Let f be any stable matching rule. Suppose by way of contradiction that there are $S' \subseteq S$ and $Q'=(Q'_{S'},Q_{-S'})$ such that $f(Q'(s)) >_{P(s)} f(Q(s))$ for every $s \in S'$. Then, $f(Q'(s)) >_{Q(s)} f(Q(s))$ by construction of $Q(s)$. But $f(Q(s))= \mu_S$, which contradicts Proposition 4* applied to $M(Q)$. Hence Q is a strong Nash equilibrium for f. Since f is arbitrary, it follows that Q is a strong Nash equilibrium in the strong sense and the proof is complete. ■

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