

ON THE CORE OF THE ONE-SIDED ASSIGNMENT GAME

by

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ABSTRACT

In the one-sided Assignment game any two agents can form a partnership. If this is done, the partners undertake some joint activity, which produces a gain that is splitted between them. We approach this model by focusing on simple outcomes - feasible and individually rational outcomes where only unmatched agents can block. We prove that this blocking can be done in such a way that the payoffs from the trades done are not changed as players reach the core. In addition, the total sum of these payoffs is the same at any core outcome. The gains in insight with this approach allows a necessary and sufficient condition for the non-emptiness of the core to be identified. Several properties of the core outcomes of economic interest are proved.

Keywords: matching, assignment game, core, Pareto optimal simple outcome

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INTRODUCTION

The matching markets have as primary object the formation of partnerships. When agents belong to two disjoint and finite sets and can pair only with agents on the opposite set, the matching market is said to have two sides. Two-sided matching markets have been widely studied by many authors since the seminal paper of Gale and Shapley (1962) and constitute an important part of the emerging field of market design (see, for example, Roth and Xioling Xing, 1994; Roth, Sonmez and Unver, 2004). Nevertheless, very little attention has been addressed to more general matching markets, where any two agents can trade. These are the so-called *one-sided matching markets*. For a wide class of such markets, if a partnership is formed then the partners undertake some joint activity, which produces a gain that is splitted between them. We may think of agents as being, for example, scientific researchers of an academic Institution, who are constrained, for some reason, to be paired among them in order to be able to engage in *exclusive* bilateral scientific projects. A contract between agents i and j specifies how the gain $a_{\{i,j\}}$ is to be divided among the partners into their individual payoffs u_i and u_j . In some markets these payoffs are pre-set and fixed, and constitute one of the factors that determine the preferences that each researcher has over their colleagues. In some other markets, the payments inside each partnership are negotiated between the partners. Traditionally, the former kind of market has been modeled as a discrete matching market, known as Roommate Problem since Gale and Shapley (1962). The latter, not yet explored in the literature, can be modeled as a one-sided matching market with linear utilities, and can be viewed as an extension of the Assignment Game of Shapley and Shubik (1972). This is the matching model of our title and the one to be treated in this paper.

An *outcome* specifies a matching plus a payoff for each agent. It is *stable* if it is feasible, individually rational and no pair of agents can negotiate a payoff, which is higher than the payoff they obtain from the current outcome. Our first investigations show that *the set of stable outcomes equals the core of this game*. In addition, *a matching is compatible with a core payoff if and only if it maximizes the gain of the grand coalition*. This result allows us, in contrast to the Roommate model, to concentrate on the payoffs to the players rather than on the underlying matching.

The question that naturally emerges from this game is if the core is always non-empty. The answer is negative. A simple example with three players, and such that the gain of any partnership is 1, illustrates a situation in which the set of stable outcomes is empty. The central issue of this paper is, therefore, the identification of a necessary and sufficient condition for the existence of core outcomes. This problem is treated by using the conceptual framework of simple outcomes. These are *feasible and individually rational outcomes, which can only be blocked by agents who are not matched*. That is, simple outcomes are feasible and individually rational outcomes in which none of the matched players is member of a blocking pair. The set of simple outcomes is non-empty, since the outcome where every one is unmatched is simple. Clearly, the stable outcomes are simple.

Simple outcomes can be interpreted as being those outcomes resulting from the interactions between the players, leading to a core outcome, when such an outcome exists. Indeed, such interactions take place in a dynamic reformulation of the pre-game that the traditional cooperative theory of von Neuman and Morgenstern postulates: Players are permitted to freely communicate, make binding agreements and preferences over outcomes, as well as the rules of the game are common knowledge. If agents "behave cooperatively", then one can expect to observe a core outcome. Under this approach, the history underlying a core outcome, as well as whether all agreements are made at once or sequentially, make no difference at all. It is a matter for the players to work out for themselves in order to get a specific agreement. The premise of cooperative behavior has the natural meaning: *Two agents will only emerge with an agreement if they believe that more favorable terms cannot be obtained elsewhere.*

In our dynamic reformulation, transactions take place at each stage and non-trading agents at some stage may become trading agents at a subsequent stage. Along this game, if agents behave cooperatively and some trading agent breaks his/her present partnership to form a new one with some other agent, both agents will be indifferent between the terms of the new and current partnerships. That is, once a transaction is done at a given stage, the agents involved will keep their payoffs (even if they dissolve their current partnerships to enter new ones) at the subsequent stages. Consequently, only agents who are not trading at a given stage are able, by trading among them at a subsequent stage, to increase their

payoffs. This means that only non-trading agents can block the outcomes that result at any stage of the game, so all these outcomes are simple.

The reason for considering a dynamic game rather than a static game is that in practice, the hypothesis that the partnerships are formed at once is a hindrance to seeing real situations, as those that occur in many of certain negotiation processes, being modeled as a static cooperative game.

The novelty here is that, in contrast with the balancedness condition, our condition emerges naturally and intuitively: If equilibrium exists, new trades can be expected at each stage, until no transaction is able to benefit the agents involved. At this point the core has been reached. This fact is independent of which trades were done. That is, every simple outcome can be extended to another simple outcome. If the core is empty, then there is some stage at which any new trade requires that some of the agents involved do not behave cooperatively. In the example mentioned above the outcome where every one is unmatched is the only simple outcome. Thus, if two players decide to form a partnership, the resulting outcome cannot be simple, so at least one of the two players is not behaving cooperatively.

In any case the intuition is correct as confirmed here:

The core is non-empty if and only if every simple and unstable outcome can be extended to a simple outcome.

That is, under the assumption of the non-emptiness of the core, given a simple and unstable outcome x , there is a simple outcome (actually, there is a stable outcome) that keeps the payoffs of the trading agents at x and increases the payoff of at least one agent. If this condition is satisfied, the core is non-empty because no simple and unstable outcome is Pareto optimal among all simple outcomes. Since such Pareto optimal outcomes always exist (the set of simple payoffs is a non-empty and compact set), this condition is also sufficient for the existence of core outcomes.

In practical terms, simple outcomes provide an economic intuition of how blocking can be done by non-trading agents when the core is non-empty. Of course, a simple outcome out of the core is blocked by at least one pair of non-trading agents. If the core is non-empty, this blocking can be done in such a way that the payoffs from the trades done need not be changed if players reach the core. In addition, as it is proved here, *the total sum of these payoffs is the same at any stable outcome.*

Our approach also allows us to capitalize on the properties of the simple outcomes to draw conclusions about the core outcomes. In developing the theory to deal with simple outcomes we prove a powerful lemma that enables us to derive all of our main results:

If (u,x) is a simple outcome and (w,y) is a stable outcome then both x and y decompose the matched players at x who have different mates at y into two disjoint sets of the same size: the set of people who prefer (u,x) to (w,y) and the set of people who prefer the opposite. The outcomes (u,x) and (w,y) map a set onto the other set.

An immediate corollary of this lemma reflects an opposition of interests between the players involved in a partnership regarding two stable outcomes:

(A.1) *If j is matched to k under (u,x) , and to h under (w,y) , and $u_j > w_j$ then $w_k > u_k$ and $w_h > u_h$.²*

The above lemma also concurs to the following new result, which concerns a set of players who are indifferent between all stable outcomes.

(A.2) *The non-trading agents under a stable outcome have payoff zero in every simple outcome, and consequently, in every stable outcome.³*

This implies that if an agent has a positive payoff under a simple outcome then he/she is engaged in some negotiation at every simple outcome.

What these two results suggest is that, while the two-sided Assignment game moves to the one-sided Assignment game, certain properties found in the core of the first game persist for the core of the later. Properties (A.1) and (A.2) are included among them. However, some other properties of the core of the first game are lost, as for example the lattice property and its non-emptiness, as well as the existence of a polarization of interests between the two sides of the market along the whole core. In this sense, properties (A.1) and (A.2), which are inherent of the core of the majority of two-sided matching markets, may even be more fundamental than previous results have suggested.

² The polarization of interests between the two sides of the Assignment game along the whole core is a restriction of this result to that market (Shapley and Shubik, 1972 and Demange and Gale, 1985).

³ This result was proved for the continuous two-sided matching model by Demange and Gale (1985) and has an analogue for the Marriage model (Gale and Sotomayor, 1985-a, McVite and Wilson, 1970) and College admissions model (Gale and Sotomayor, 1985-b and Roth, A., 1984).

We believe that the theory of simple outcomes developed here opens a new way to deal with the existence problem of core outcomes of more general coalitional games, with the additional advantage of providing economic intuition of the results and simple proofs.

This paper is organized as follows. In section 2 we describe the model and present the preliminary results and definitions. Section 3 introduces the concept of simple outcome, proves some of its properties and gives the conceptual framework to be used in section 4. Section 4 is devoted to the existence and non-existence of stable outcomes. Section 5 concludes the paper and discusses related work.

2. THE FORMAL MODEL AND SOME PRELIMINARIES

There is a finite set of players, $N=\{1,2,\dots,n\}$. Associated to each partnership $\{i,j\}$ there is a nonnegative real number $a_{\{i,j\}}$ which will be denoted a_{ij} . A game in coalitional function form with side payments is determined by (N,a) , with the numbers a_{ij} being equal to the worth of the coalitions $\{i,j\}$. The worth of large coalitions is determined entirely by the worth of the pairwise combinations that the coalition members can form. That is, the coalitional function v is given by

$$v(S)=0 \text{ if } |S|=1;$$

$$v(S)=a_{ij} \text{ if } S=\{i,j\};$$

$$v(S)=\max\{v(i_1,j_1)+v(i_2,j_2)+\dots+v(i_k,j_k)\}^4 \text{ for arbitrary coalitions } S, \text{ where } k \text{ is an integer number that does not exceed the integer part of } |S|/2. \text{ The maximum is taken over all sets } \{i_1,j_1\},\dots,\{i_k,j_k\} \text{ of } k \text{ distinct pairs in } S.$$

Thus, the rules of the game are that any pair of agents $\{i,j\}$ can together obtain a_{ij} , and any larger coalition is valuable only insofar as it can organize itself into such pairs. The members of any coalition may divide among themselves their collective worth in any way they like. We might think of the two-sided Assignment game of Shapley and Shubik (1972) as being a particular case of this game, by taking $N=P\cup Q$, $P\cap Q=\phi$, $v(S)=0$ if S contains only agents of P or only agents of Q .

We will consider that any player i can be self-matched and will define $a_{ii}=0$ for all $i\in N$. Then,

⁴ We write $v(i,j)$ rather than $v(\{i,j\})$.

Definition 1. A *feasible matching* x is a one-to-one correspondence from N onto itself of order two (that is, $x^2(j)=j$). We refer to $x(j)$ as the partner of j at x . If $x(i)=i$ we say that i is **unmatched** at x .

For our purposes it is simpler to define,

Definition 2. The vector u , with $u \in R^n$, is called a *feasible payoff* for (N,a) if there is a feasible matching x such that

$$u_i + u_j = a_{ij} \text{ if } x(i)=j \text{ and } u_i=0 \text{ if } x(i)=i.$$

In this case we say that (u,x) is a *feasible outcome* and x is compatible with u .

Remark 1. Given a coalition S , the definition of v implies that there is some feasible matching x such that $x(S)=S$ and $\sum_{i \in S, i \leq x(i)} a_{i,x(i)} = v(S)$. Furthermore, $v(S) \geq \sum_{i \in S, i \leq x'(i)} a_{i,x'(i)}$ for all feasible matching x' such that $x'(S)=S$. Then, it follows from Definition 2 that $\sum_{i \in S} u_i \leq v(S)$ for all $S \subseteq N$ and feasible outcome (u,x) with $x(S)=S$. In particular, $\sum_{i \in N} u_i \leq v(N)$.

The key definition is the following:

Definition 3. The feasible payoff u is *stable* if

- (i) $u_i \geq 0$, for all $i \in N$,
- (ii) $u_i + u_j \geq a_{ij}$ for all $\{i,j\} \subseteq N$.

If x is compatible with u we say that (u,x) is a *stable outcome*.

Condition (i) (individual rationality) means that a player always has the option of remaining unmatched. Condition (ii) is the natural one: If it is not satisfied for some agents i and j , then it would pay them to break up their present partnership(s) and form a new one together, because this could give them each a higher payoff. In this case, we say that $\{i,j\}$ blocks u .

The following example shows that the set of stable outcomes may be empty.

Example 1. Consider $N=\{1,2,3\}$ and $a_{ij}=1$ for all $\{i,j\}\subseteq N$. For every feasible payoff u there will exist two players i and j such that $u_i+u_j<1$. Hence, the set of stable outcomes of this game is empty.■

Definition 4. The payoff u is in the **core** of (N,a) if $\sum_{i\in N}u_i=v(N)$ and $\sum_{i\in S}u_i\geq v(S)$ for all $S\subseteq N$.

Proposition 1. The set of stable payoffs equals the core of (N,a) .

Proof. Suppose u is a stable payoff. Then, u is feasible and so

$$\sum_{i\in N}u_i\leq v(N), \quad (1)$$

by Remark 1. Given a coalition S , let y be a feasible matching such that $y(S)=S$ and $v(S)=\sum_{i\in S, i\leq y(i)}a_{\{i,y(i)\}}$. The stability of u implies that $u_i+u_{y(i)}\geq a_{i,y(i)}$ for all $i\in S$, so

$$\sum_{i\in S}u_i\geq v(S) \text{ for all coalition } S. \quad (2)$$

By (1) and (2) it follows that $\sum_{i\in N}u_i=v(N)$ and $\sum_{i\in S}u_i\geq v(S)$ for all $S\subseteq N$, so u is in the core.

Now, suppose u is in the core. Definition 4 implies that $u_i+u_j\geq v(i,j)=a_{ij}$ for every coalition $\{i,j\}$ and $u_i\geq v(i)=0$ for all $i\in N$, so u does not have any blocking pair and is individually rational. To see that u is feasible let x be a feasible matching such that $v(N)=\sum_{i\leq x(i)}a_{\{i,x(i)\}}$. Now use that $\sum_{i\in N}u_i=v(N)$ and $u_i+u_{x(i)}\geq a_{i,x(i)}$ for all $i\in N$, to get that $\sum_{i\in N}u_i=\sum_{i\leq x(i)}a_{\{i,x(i)\}}\leq\sum_{i\leq x(i)}(u_i+u_{x(i)})=\sum_{i\in N}u_i$, so the inequality cannot be strict and so $u_i+u_{x(i)}=a_{i,x(i)}$ for all $i\in N$. Since $u_i\geq 0$, it follows that $u_i=0$ if $x(i)=i$. Hence u is stable and the proof is complete.■

Definition 5. The feasible matching x is optimal if $\sum_{i\leq x(i)}a_{\{i,x(i)\}}=v(N)$.

The following two propositions make clear why, similarly to the two-sided Assignment game and in contrast to the discrete version (roommate-problem), we can concentrate on the payoffs to the agents rather than on the underlying matching.

Proposition 2. If x is an optimal matching, then it is compatible with any stable payoff u .

Proof. Immediate from the fact that if u is a stable payoff then $u_i + u_{x(i)} \geq a_{i,x(i)}$ for all $i \in N$, so $v(N) = \sum_{i \in N} u_i = \sum_{i \in x(i)} (u_i + u_{x(i)}) \geq \sum_{i \in x(i)} a_{i,x(i)} = v(N)$, where the last equality is implied by the optimality of x . But then, the inequality cannot be strict and $u_i + u_{x(i)} = a_{i,x(i)}$ for all $i \in N$. Since $u_i \geq 0$, it follows that $u_i = 0$ if $x(i) = i$. Hence, x is compatible with u . ■

Proposition 3. *If (u, x) is a stable outcome, then x is an optimal matching.*

Proof. Immediate from the fact that

$$v(N) = \sum_{i \in N} u_i = \sum_{i \in x(i)} (u_i + u_{x(i)}) = \sum_{i \in x(i)} a_{i,x(i)}. \blacksquare$$

3. SIMPLE OUTCOMES: CONCEPTUAL FRAMEWORK AND PROPERTIES

The feasible and individually rational outcome (u, x) is **simple** if every player is unmatched at x or, in case $x(i) \neq i$ for some i , then i is not part of any blocking pair of (u, x) . Hence, in case a blocking pair $\{i, j\}$ exists, i and j are unmatched at x . Since the outcome where everyone is unmatched is simple, **the set of simple outcomes is non-empty**. Clearly, every stable outcome is simple. If (u, x) is a simple outcome we say that u is a simple payoff.

As mentioned in section 1, the key lemma is:

Lemma 4. *Let (u, x) be a simple outcome and let (w, y) be a stable outcome. Let $T = \{j \in N; x(j) \neq j\}$, $M_u = \{j \in N; u_j > w_j\}$, $M_w = \{j \in T; w_j > u_j\}$ and $M_0 = \{j \in T; u_j = w_j\}$. Then $x(M_u) = y(M_u) = M_w$ and $x(M_w) = y(M_w) = M_u$. Furthermore, $x(M_0) = M_0$.*

Proof. All j in M_u are matched under x , since $u_j > w_j \geq 0$. Analogously, all j in M_w are matched under y , since $w_j > u_j \geq 0$. If j is in M_u then $k = x(j)$ is in M_w , for if not

$$a_{jk} = u_j + u_k > w_j + w_k$$

which implies that (j, k) blocks (w, y) , contradiction. On the other hand, if k is in M_w then $j = y(k)$ is in M_u , for if not

$$a_{jk} = w_j + w_k > u_j + u_k$$

which implies that (j, k) blocks x . However, k is in T , so k is matched under x , which contradicts the fact that (u, x) is simple. Therefore, $x(M_u) \subseteq M_w$ and $y(M_w) \subseteq M_u$, so $M_u \subseteq x(M_w)$ and $M_w \subseteq y(M_u)$. It follows that

$$|M_u| = |x(M_u)| \leq |M_w| = |y(M_w)| \leq |M_u| \quad \text{and} \quad |M_w| \leq |y(M_u)| = |M_u| \leq |x(M_w)| = |M_w|,$$

which implies $x(M_u)=M_w$, $y(M_w)=M_u$, $y(M_u)=M_w$ and $x(M_w)=M_u$. The last part of the lemma follows from the fact that $x(M_0)\subseteq M_0$ and x is one-to-one. Hence the proof is complete. ■

An immediate consequence of this lemma is

Proposition 5. *Let (u,x) and (w,y) be stable outcomes. If j is matched to k under x or under y and $u_j > w_j$ then $w_k > u_k$.*

If j has a positive payoff under a simple outcome (u,x) then he/she is matched under every stable outcome. In fact,

Proposition 6. *Let (u,x) be a simple outcome and let (w,y) be a stable outcome. If j is unmatched under y , $u_j = 0$.*

Proof. Suppose j is unmatched under y . If $u_j > 0 = w_j$ define M_u as in Lemma 4. Then $j \in M_u$, so Lemma 4 implies j is matched under y , which is a contradiction. ■

The fact that every stable outcome is simple implies that **if j is unmatched under a stable outcome then he/she gets payoff zero under any stable payoff.**⁵

In what follows we will make use of the following definition:

Definition 6. *Let (u,x) be a simple outcome. Let $T = \{j \in N; x(j) \neq j\}$. We say that the feasible outcome (u^*,z) extends (u,x) if $u^*_j > u_j$ for some $j \notin T$ and $u_j = u^*_j$ for all $j \in T$. If (u^*,z) is simple (respectively stable) then (u^*,z) is said to be a simple (respectively stable) extension of (u,x) .*

Given a simple outcome (u,x) , which is not in the core, we can always obtain a new outcome (u^*,z) that keeps the payoffs of the matched players. What is new is that the outcome (u^*,z) can be constructed so that it is stable.

⁵ This result was proved in Demange and Gale (1985) for a two-sided matching market where the utilities are continuous, so it applies to the two-sided Assignment game of Shapley and Shubik (1972).

Proposition 7. *Let (u,x) be an unstable and simple outcome. Suppose the set of stable outcomes is non-empty. Then there exists a stable outcome (u^*,z) that extends (u,x) .*

Proof. Let (w,y) be a stable outcome. Define M_u , M_w and M_0 as in Lemma 4. Let $T=\{j \in N; x(j) \neq j\}$. Then $T=M_u \cup M_w \cup M_0$. Now construct the outcome (u^*,z) as follows: $z(j)=x(j)$ and $u^*_j=u_j$ if $j \in M_u \cup M_w$; $z(j)=y(j)$ and $u^*_j=w_j$ otherwise. It follows from Lemma 4 that all of $M_u \cup M_w$ are matched among themselves under y , so z is feasible. We are going to show that

$$u^*_j \geq u_j \text{ for all } j \in N; \quad (1)$$

$$\text{and } u^*_j = u_j \text{ for all } j \in T. \quad (2)$$

In fact, if $j \notin M_u \cup M_w$ then $j \in M_0$ or $x(j)=j$. In the first case, $u_j=w_j=u^*_j$ and in the other case $u_j=0 \leq w_j=u^*_j$. Therefore, $u^*_j=u_j$ for all $j \in T$ and $u^*_j \geq u_j$ for all $j \in N$. We claim that (u^*,z) is stable. That (u^*,z) is feasible and individually rational is immediate from the feasibility and individual rationality of (u,x) and (w,y) . Thus, it remains to show that (u^*,z) does not have any blocking pair. The fact that (u,x) is simple and (w,y) is stable implies that $\{j,k\}$ does not block (u^*,z) in the following cases: $\{j,k\} \subseteq M_u \cup M_w$ and $\{j,k\} \subseteq N-[M_u \cup M_w]$. Then, without loss of generality, suppose $j \in M_u \cup M_w$ and $k \in N-[M_u \cup M_w]$. If $\{j,k\}$ blocks (u^*,z) , we must have $a_{jk} > u^*_j + u^*_k \geq u_j + u_k$, where in the last inequality we used (1). In this case (j,k) would block (u,x) . However, $j \in M_u \cup M_w$, so j is matched at x , which contradicts the fact that (u,x) is simple. Hence, in any case, (u^*,z) does not have any blocking pair, so it is stable.

To see that (u^*,z) extends (u,x) , use (1) and (2) to conclude that $u^*_i \geq u_i$ for all $i \in N$. The fact that (u^*,z) is stable, (u,x) is unstable and $u^*_j = u_j$ for all $j \in T$ implies that there is some $j \notin T$ such that $u^*_j > u_j$. Hence the proof is complete. ■

The following proposition asserts that the sum of the payoffs of the matched agents is the same under every stable outcome.

Proposition 8. *Let (u,x) be a simple outcome and let (w,y) be in the core. Let $T=\{j \in N; x(j) \neq j\}$. Then, $\sum_{i \in T} u_i = \sum_{i \in T} w_i$.*

Proof. It follows from Remark 1 and from the fact that T does not block (u,x) that $\sum_{i \in T} u_i = v(T)$. Define the stable outcome (u^*,z) as in the proof of Proposition 7. Then, $v(N) = \sum_{i \in N} u^*_i = \sum_{i \in T} u_i + \sum_{i \in N-T} w_i = v(T) + \sum_{i \in N-T} w_i \leq \sum_{i \in T} w_i + \sum_{i \in N-T} w_i = \sum_{i \in N} w_i = v(N)$, so $v(T) = \sum_{i \in T} w_i$, where in the inequality was used that T does not block (w,y) . Hence, $\sum_{i \in T} u_i = \sum_{i \in T} w_i$, and the proof is complete. ■

4. EXISTENCE OF STABLE MATCHINGS

Theorem 10 asserts that the condition that every simple and unstable outcome has a simple extension is necessary and sufficient for the non-emptiness of the core. We need one more concept.

Definition 7. *The payoff u is a **Pareto optimal simple payoff (PS for short)** if it is simple and there is no simple payoff w such that:*

- (i) $w_j \geq u_j$ for all players j and
- (ii) $w_j > u_j$ for at least one player j .

If x is compatible with u , we say that (u,x) is a Pareto optimal simple outcome.

Therefore, if u is PS and $w_j > u_j$ for some player j and some simple payoff w , there is some other player k such that $w_k < u_k$. Proposition 9 asserts that the set of simple payoffs is a compact set of R^n . Then, there is some simple payoff u^* such that $\sum_{j \in N} u^*_j \geq \sum_{j \in N} u_j$ for all simple payoffs u . Clearly, u^* is a Pareto optimal simple payoff.

Proposition 9. *The set of simple payoffs is a compact set of R^n .*

Proof. The set of simple payoffs is bounded, since $0 \leq u_j \leq v(N)$, for all $j \in N$ and all simple payoff u . To see that it is closed, take any sequence $(u^t)_t$ of simple payoffs, with $u^t \rightarrow u$, when t tends to infinity. Since the set of matchings is finite, there is some matching x , which is compatible with infinitely many terms of the sequence u^t . We will use the same notation (u^t) for this subsequence. Then, if $x(j)=k$, $u_j + u_k = \lim_{t \rightarrow \infty} (u^t_j + u^t_k) = \lim_{t \rightarrow \infty} a_{jk} = a_{jk}$; if $x(j)=j$ then $u_j = \lim_{t \rightarrow \infty} u^t_j = 0$. Thus, x is compatible with u , so (u,x) is feasible. Now, observe that if j is matched at x then j is not part of a blocking pair of

(u,x) . In fact, $u_j + u_k = \lim_{t \rightarrow \infty} (u_j^t + u_k^t) \geq \lim_{t \rightarrow \infty} a_{jk} = a_{jk}$. Therefore, (u,x) is simple, so u is a simple payoff. Hence, the set of simple payoffs is bounded and closed, so it is compact. ■

We can now prove our main result.

Theorem 10. *The set of stable outcomes is non-empty if and only if every unstable and simple outcome has a simple extension.*

Proof. If the set of stable outcomes is non-empty the result follows immediately from Proposition 7. In the other direction, let (u,x) be a Pareto optimal simple outcome. We are going to show that (u,x) is stable. In fact, suppose by way of contradiction that (u,x) is unstable. By hypothesis, there is some simple outcome (u^*,z) , which extends (u,x) . Then, $u_j^* \geq u_j$ for all $j \in N$ and $u_j^* > u_j$ for at least one player j . But this contradicts the fact that (u,x) is a PS outcome. Hence (u,x) is stable and the proof is complete. ■

5. CONCLUDING REMARKS AND RELATED WORK

This paper deals with a new generalization of the two-sided Assignment game of Shapley and Shubik (1972)⁶ to the case where any two agents can form a partnership. It provides new results and a new point of view through the concepts of simple outcome and Pareto optimal simple outcome. Stable outcomes are simple and Pareto optimal, but not all Pareto optimal outcomes are stable or Pareto optimal simple. Unstable outcomes may be Pareto optimal and unstable simple outcomes may be Pareto optimal simple. However, we proved that the core is non-empty if and only if no unstable simple outcome is Pareto optimal simple.

The present study represents part of a continuing investigation of the properties of the simple outcomes aiming to deal with the existence problem of core outcomes. Instead of the conventional "static game", which does not take into account the way players reach the core, we postulate a finite "dynamic game", that is, a finite sequence of simple and unstable outcomes. At each stage, if it is possible for some group of agents to "interact",

⁶ You can also see Roth and Sotomayor (1990) where an overview of the two-sided Assignment game is presented.

that is, to agree about their payoffs under the premise of cooperative behavior, then a simple outcome results. The game ends when no interaction is able to benefit the agents involved or when any new interaction requires that some of the agents involved do not behave cooperatively. In the first case a core outcome is reached and in the other case the core is empty.

The reason for considering a dynamic game rather than a static game is quite simply that the former is what in reality we have, the latter what we have not. Of course, every economic model involves some compromises with reality, and the static game is probably an acceptable compromise for the study of the properties of the core outcomes. But for other purposes, in particular for the existence or non-existence of core outcomes, this paper showed that our results emerge naturally and intuitively from the dynamic game: when the core is non-empty, each particular core outcome is associated to a sequence of simple outcomes produced at each stage of the game. The particular core outcome is reached at the end of the game and naturally extends any simple outcome of the sequence. Of course, if the game ends without reaching the core then the core is empty.

The practical advantage of this new approach for the one-sided Assignment game is that it enabled us to make elementary proofs by only using simple combinatorial arguments, rather than the Linear Programming theorems commonly used in the models with linear utilities (see Shapley and Shubik, 1972 and Sotomayor, 1992).

The theory of simple outcomes has been used in several other models. Its main feature has been to allow that results be proved without the framework of sophisticated mathematical tools. A natural adaptation of the concept of simple outcome was introduced in Sotomayor (2005-a) and in Sotomayor (2005-b). The former paper proves that the stability condition is also necessary and sufficient for the non-emptiness of the core of the Roommate model of Gale and Shapley (1962), the non-transferable utility version of our game. In the latter, this condition is proved to be always satisfied for the Housing market with strict preferences of Shapley and Scarf (1974); so the core is always non-empty in this model.

As for two-sided markets, a version of the concept of simple outcome has been introduced in: a) Sotomayor (1996), for the Marriage market of Gale and Shapley (1962); b) Sotomayor (1999), for the College Admission model of Gale and Shapley (1962) and the

discrete many-to-many matching model with substitutable and non-strict preferences; and in c) Sotomayor (2000), for the two-sided Assignment game of Shapley and Shubik (1972) and the unified two-sided matching model of Eriksson and Karlander (2000). In each of these models a non-constructive existence proof of pairwise-stable outcomes has been provided by showing that simple outcomes that are pairwise unstable are not Pareto optimal simple outcomes.

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