

**CONNECTING THE COOPERATIVE AND COMPETITIVE STRUCTURES
OF THE MULTIPLE-PARTNERS ASSIGNMENT GAME**

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ABSTRACT

Multiple-partners assignment game is the name used by Sotomayor [22] and [23] to describe the cooperative structure of the many-to-many matching market with additively separable utilities. Stability concept is proved to be different from the core concept. An economic structure is proposed where the concept of competitive equilibrium payoff is introduced in connection to the equilibrium concept from standard microeconomic theory. The paper examines how this equilibrium concept compares with the cooperative equilibrium concept. Properties of interest to the cooperative and competitive markets are derived.

Keywords: matching, setwise-stability, core, competitive equilibrium, lattice.

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INTRODUCTION

Labor markets in which agents are firms and workers are usually endowed with a cooperative game structure, while buyers and sellers markets have a competitive market game structure. The link between these two game structures, in the same environment, is the central issue of this paper. More specifically, we want to know how the competitive equilibrium concept, closely related to the traditional concept of equilibrium from standard microeconomic theory, compares with the cooperative equilibrium concept that in this class of games is known as stability.

For the one-to-one buyer-seller market game, also known as Assignment Game, Shapley and Shubik [21] proved that the core is a non-empty complete lattice that coincides with the set of stable payoffs and with the set of competitive equilibrium payoffs. We restrict attention to a more encompassing model that generalizes the Assignment Game to a many-to-many matching model with additively separable utilities. (See Crawford and Knoer [4] and Sotomayor [22] and [23]). In this game there are two finite and disjoint sets of players, B and Q . A player of a set may form more than one partnership with different players of the other set. Every participant has a quota representing the maximum number of partners. The main characterization of this game is that players are able to negotiate their individual payoffs: If b and q become partners, they undertake an activity together that produces a gain v_{bq} , which is divided between them the way both agree: $u_{bq} \geq 0$ for b and $w_{bq} = v_{bq} - u_{bq} \geq 0$ for q . Therefore, an outcome for this game is a matching, that is, any set of partnerships that does not violate the quotas of the players, along with individual payoffs

u_{bq} 's and w_{bq} 's.² The Assignment Game is a special case, where the only restriction is that each agent is allowed to form one partnership at most.

We see two different ways of interpreting this many-to-many assignment game. The first one considers a labor market of firms and workers operating cooperatively. That is, we postulate an environment where agents can communicate with each other, and offers and counter offers are made, culminating with a set of bilateral and exclusive contracts. The cooperative behavior supposes that a contract between a firm and a worker will not be signed unless both agents are sure that more favorable terms cannot be obtained elsewhere. The quota of a firm is the maximum number of workers it can hire; the quota of a worker is the maximum number of jobs he/she can take and the number v_{bq} is the productivity of worker q in firm b . The appropriate equilibrium concept is the one of setwise-stability (stability for short), shown here to be different from the core concept and equivalent to the concept of pairwise-stability. Of course, the core always contains the set of stable payoffs. However, an example in this paper demonstrates that there may be core payoffs that are not stable, so the core may be bigger than the set of stable payoffs.³ The equivalence with pairwise-stability implies that, although coalitions of many players matched among them may be formed, to check instabilities we only need to consider two-agent coalitions made up of agents on opposite sides. Thus, an outcome x will be called *stable* unless there are agents b and q that do not form a partnership at x , but that can increase their total payoff, becoming partners and at the same time keeping and/or leaving some of their old partners if necessary, in order to remain within their quotas.

² Another approach of the present model is treated in Sotomayor [22]. There, agents are not allowed to negotiate their individual payments and act in blocks. An outcome only specifies their total payoffs. The core coincides with the set of stable payoffs and is not a lattice. Pairwise-stability is not equivalent to corewise-stability.

The other interpretation approaches the game competitively through a buyer-seller market: B is a set of buyers and Q is a set of sellers. Buyers are interested in sets of objects owned by different sellers and each seller owns a lot of identical objects. Given the prices of the objects, buyers demand their favorite sets of objects. The quota of a buyer is the number of objects she is allowed to acquire; the quota of a seller is the number of identical objects he owns and the number v_{bq} is the amount of money buyer b considers to pay for an object of seller q . The equilibrium concept is called *competitive equilibrium payoff*. Roughly speaking, the payoff $(u, w; \mu)$ is a *competitive equilibrium payoff* if μ is a feasible allocation, and given prices w , each active buyer receives one of her demand sets of items (i.e. a set of items that, given prices, maximizes her additive utility payoff), every inactive buyer has payoff zero and every unsold object has zero price. (The formal definition is given in the text). A consequence of this concept is that sellers do not discriminate buyers under a competitive equilibrium outcome, as they might do under a stable outcome. That is, every seller sells all his objects at the same competitive equilibrium price.

This paper examines the connection between the set of stable payoffs and the set of competitive equilibrium payoffs, by focusing on the algebraic structures of these two sets. We show that the set of competitive equilibrium payoffs is non-empty and is endowed with a complete lattice structure under convenient partial order relations, \geq_B and \geq_Q , defined in the text. There is a conflict of interests between the two sides of the market with respect to two comparable competitive equilibrium payoffs, x and y : The best competitive equilibrium payoff for one side is the worst competitive equilibrium payoff for the other

³ Core allocations are identified with stable allocations for the one-to-one and many-to-one restrictions of this model. See Sotomayor [27].

side. In addition, due to the fact that sellers do not discriminate buyers under a competitive equilibrium outcome, the preferences of the buyers are opposed to those of the sellers along the whole set of competitive equilibrium payoffs. Although the partial orders \geq_B and \geq_Q are not defined by the preferences of the players, it turns out that all players in B , as well as all players in Q , agree on the best competitive equilibrium payoff for them. These outcomes are called B -optimal competitive equilibrium payoff and Q -optimal competitive equilibrium payoff, respectively. Due to the polarization of interests, the B -optimal competitive equilibrium payoff (respectively, Q -optimal competitive equilibrium payoff) is the worst competitive equilibrium payoff under the point of view of the players in Q (respectively, P).⁴

These results are similar to a characterization of the set of stable payoffs, as obtained in Sotomayor [23]. However, unlike the one-to-one case, the lattice of the stable payoffs is different from the lattice of the competitive equilibrium payoffs. In fact, our main finding is that the lattice of competitive equilibrium payoffs can be obtained by "shrinking" the lattice of stable payoffs through the application of an *isotone* (order-preserving) and non-identical map f . Thus, the polarization character of the set of stable payoffs remains. The competitive equilibrium payoffs are exactly the fixed points of f . The B -optimal and the Q -optimal stable payoffs are mapped to the B -optimal and the Q -optimal competitive equilibrium payoffs, respectively. However, there are stable payoffs that are not fixed points, and so there are stable payoffs that are not competitive. For each stable outcome the function reduces the total payoff to every seller by reducing the price of each of his items to his minimal individual payoff, to create a competitive equilibrium. It turns out that this

⁴ Shapley and Shubik [21] prove these results for the one-to-one Assignment Game. See also Roth and Sotomayor [20] for an overview of this model.

resulting outcome is still stable, so the worst stable payoff for sellers must be competitive, and so the B -optimal stable payoff equals the B -optimal competitive equilibrium payoff.

These results are tied together by the lattice structure of the set of stable payoffs and follow from a powerful *algebraic* fixed-point theorem due to Alfred Tarski (see in the text).

The connection between the core, the set of stable payoffs and the set of competitive equilibrium payoffs is illustrated graphically via an example. This picture shows that the core is bigger than the set of stable payoffs, which in its turn is bigger than the set of competitive equilibrium payoffs. Unlike the one-to-one and the many-to-one matching models, the core is not endowed with a lattice structure⁵. Also the polarization of interests with respect to the two extreme points of the lattice fails to hold for the core payoffs: The best core payoff for players in B is not the worst core payoff for players in Q . Indeed, there is no such core payoff.

This paper is organized as follows. In section 2 we describe the cooperative and competitive structure. Sub-section 2.1 concerns the competitive behavior of the agents. Section 3 discusses the conclusions of the main results through a graphic example. Section 4 focuses on the lattice property of the set of competitive equilibrium payoffs. Section 5 discusses the role played by quotas, additive separability, rules of the game and multiplicity of partnerships in the choice of a cooperative equilibrium concept and in the main results of the paper. Section 6 concludes the paper and presents some related work. Some of the proofs are presented in the Appendix.

2.THE FRAMEWORK

The many-to-many matching model discussed in the Introduction consists of two finite and disjoint sets of players, B and Q , which we can think of as being buyers and sellers, respectively. The B -players may form more than one partnership with Q -players, and Q -players may form more than one partnership with B -players. The set B has m elements and the set Q has n elements. Each $q \in Q$ has a quota $s(q)$ and each $b \in B$ has a quota $r(b)$, representing the maximum number of partnerships they can form. Quota $s(q)$ of seller q means that q owns $s(q)$ identical and indivisible objects, and quota $r(b)$ of buyer b represents the maximum number of objects buyer b is allowed to buy. Without loss of generality we can consider $r(b) \leq n$ and $s(q) \leq m$. No buyer is interested in acquiring more than one item of a given seller. In section 5 the quota of a player is endogenously defined.

Generically, we will denote buyers by b, b' , and sellers and objects by q, q' . Every object has a reservation price of 0 (*which can be obtained after normalization*). For each pair (b, q) there is a non-negative number $v_{bq} \geq 0$ which is splitted between b and q if both form a partnership. We can interpret this number as the value of any object of seller q to buyer b . That is, v_{bq} is the gain of trade when any of the objects of seller q is sold to buyer b . If buyer b acquires some object of q at price π then b receives the individual payoff $u_{bq} \equiv v_{bq} - \pi$. Dummy players, denoted by 0 , are included for technical convenience in both sides of the market. We have that $v_{b0} = v_{0q} = 0$ for all $b \in B$ and $q \in Q$. As for the quotas, a dummy player may form as many partnerships as needed to fill up the quotas of the non-dummy players. We will also include an artificial "null-object", 0 ,

⁵ That the core of the many-to-one case with separable and additive utilities is a complete lattice follows from the facts that the core and the set of stable payoffs coincide (Sotomayor [27]) and the set of stable payoffs for this model is a complete lattice (Sotomayor [23]).

owned by the artificial seller, whose value is zero to all buyers and whose price is always zero.

We will say that a subset $S \subseteq Q$ is an **allowable set of partners for $b \in B$** , if $|S| = r(b)$. We will extend this terminology to include the sets S with k non-dummy players and $r(b)-k$ repetitions of the dummy player for $0 \leq k \leq r(b)$. Analogously, we define an **allowable set of partners for $q \in Q$** . In order to simplify notation, we will also write $S \subseteq B$ or $S \subseteq Q$ for any allowable set S of B -players or Q -players, respectively. **An allowable set of objects for buyer b contains $r(b)$ objects**, some of which may be repetitions of the null-object. Furthermore, it does not contain more than one object of the same seller (an exception is made to the fictitious seller).

Under the cooperative approach, agents b and q may form a partnership and negotiate their individual payments u_{bq} and w_{bq} the way they like. A **matching** μ is a set of partnerships of the kind (b,q) , $(b,0)$ or $(0,q)$, for $(b,q) \in B \times Q$. If b and q are matched under μ , we write $b \in \mu(q)$ or $q \in \mu(b)$. A dummy player may be matched to more than one player of the opposite side and more than once to the same player.

We define a **matching μ to be feasible if each player is matched to an allowable set of partners**. The value of μ is $\sum_{q \in Q, b \in \mu(q)} v_{bq}$. The matching μ is **optimal** if it attains the maximum value among all feasible matchings.

We will be assuming that agents' preferences over potential partners are *separable across pairs*, in the sense that the payoff of the partnership (b,q) , v_{bq} , does not depend on which other partnerships are formed by buyer b and seller q . For any sets $R \subseteq B$ and

$S \subseteq Q$, the payoff $P(R \cup S)$ of coalition $R \cup S$ is the maximum $\sum_{b \in R, \mu(b) \in S} v_{b\mu(b)}$ over all feasible matchings μ .

If S is an allowable set of partners for $b \in B$, **the payoff** $P(\{b\} \cup S)$ **of the coalition** $\{b\} \cup S$ is the sum of the numbers v_{bq} 's with $q \in S$. Similarly, we define **the payoff** $P(\{q\} \cup S)$, where S is an allowable set of partners for $q \in Q$. In this paper we will refer to functions $P(\{b\} \cup \{.\})$ and $P(\{q\} \cup \{.\})$ as being *additively separable*.

This cooperative game will be denoted by M and is more appropriate to model a job market of firms and workers.⁶ In this case the numbers v_{bq} 's can be interpreted as being the productivity of worker q in firm b , which is divided between the agents into salary w_{bq} and net profit u_{bq} .

For our purposes we define:

Definition 1. *A feasible outcome for M , denoted by $(u, w; \mu)$, is a feasible matching μ and a pair of payoffs (u, w) , where the individual payoffs of each $b \in B$ and $q \in Q$ are given by the arrays of numbers u_{bq} , with $q \in \mu(b)$, and w_{bq} , with $b \in \mu(q)$, respectively, such that $u_{bq} + w_{bq} = v_{bq}$, $u_{bq} \geq 0$ and $w_{bq} \geq 0$. Consequently, $u_{b0} = u_{0q} = w_{b0} = w_{0q} = 0$ in case these payoffs are defined.*

If $(u, w; \mu)$ is a feasible outcome, we say that μ is compatible with payoff (u, w) and vice-versa.

⁶ This model was introduced in Sotomayor [22] and has also been treated in Sotomayor [23]. It is a version of the model of Crawford and Knower [4] and is the simplest extension of the Assignment game of Shapley and Shubik [21] to the case of multiple partners.

Under the competitive approach, a **feasible outcome** is a feasible allocation of the objects to the buyers plus a non-negative price for each object and a corresponding array of non-negative individual payoffs for each buyer. **A feasible allocation allocates each non-null object to one buyer** (who might be the dummy buyer) so that **each non-dummy buyer is assigned to an allowable set of objects for her**. If an object is allocated to the dummy buyer we say that it is left unsold. Of course, a dummy buyer may be assigned to any number of objects and the null object may be allocated to any number of buyers.

If an object is allocated to a buyer then the seller who owns this object is matched to that buyer. Thus, if μ^* is a feasible allocation, we can define a corresponding matching μ such that seller $q \in \mu(b)$ if and only if one of his objects is allocated to b under μ^* ⁷. We say that μ and μ^* **correspond** to each other. Clearly, μ and μ^* have the same value, so **μ is an optimal matching if and only if μ^* is an optimal allocation.**

A vector of prices $p \in R_+^N$, with $N \equiv \sum_{q \in Q} s(q)$, is called a **price vector**. Given a price vector, the preference of a buyer over sets of objects is defined only over allowable sets of objects. Due to the characteristics of the market, there is no loss by assuming that. **The value of an allowable set of objects S to buyer b** is the sum of the values of the objects in S to b . That is, it is the payoff of the coalition consisting of b and the sellers who own the objects in S . Then, given a price vector, the preferences of buyers over objects are completely described by the numbers v_{bq} 's: For any two allowable sets of objects S and S' , buyer b prefers S to S' at prices p if her total payoff when she buys S is greater than her total payoff when she buys S' . She is indifferent between these two

⁷ If μ^* is a feasible allocation and a buyer buys an object from a seller, then make a link between the two. The resulting graph is the corresponding feasible matching μ . Conversely, if μ is a feasible matching and a

sets if she gets the same total payoff with both sets. Object q is **acceptable** to buyer b at prices p if, at these prices, b likes q at least as well as the null-object.

Under the structure of preferences we are assuming, each buyer b is able to determine which allowable sets of objects she would most prefer to buy at a given price vector p . We denote the set of all such allowable sets by $D_b(p)$ and call it the **demand set of b at prices p** . (Note that $D_b(p)$ is never empty, because there is always the option of buying $r(b)$ copies of the null object. Note also that, if $S \in D_b(p)$, then every element of S is acceptable to b). This economic structure will be denoted by M^* .

The concepts of corewise-stability and pairwise-stability are usual equilibrium concepts for a variety of cooperative games in the coalitional form.

*A feasible payoff x is **corewise-stable** if there is no coalition of players who by forming **all** their partnerships only among themselves, can all obtain a higher payoff than the one given by x .*

*A feasible payoff x is **pairwise-stable** if there are no agents b and q who are not partners, but by becoming partners, possibly dissolving some of their partnerships given by x to remain within their quotas and possibly keeping other ones, can both obtain a higher payoff than the one given by x .*

The formal definitions are given below. We will use the following notation: Given a feasible payoff (u, w) , $u_b(\min)$ is the **smallest individual payoff of buyer b** ; $w_q(\min)$ is the **smallest individual payoff of seller q** ; U_b denotes b 's **total payoff**, that is, the **sum of b 's individual payoffs under (u, w)** . Similarly, W_q denotes q 's **total payoff under (u, w)** .

buyer buys an object from a seller, then make a link between the buyer and the object. The resulting graph is the corresponding feasible allocation μ^* .

Definition 2. *The feasible outcome for M , $(u, w; \mu)$, is **corewise-stable** if for all coalitions $R \subseteq P$ and $S \subseteq Q$, we have $\sum_{b \in S} U_b + \sum_{q \in R} W_q \geq P(R \cup S)$.*

Definition 3. *The feasible outcome for M , $(u, w; \mu)$, is **pairwise-stable** if $u_b(\min) + w_q(\min) \geq v_{bq}$ for all pairs (b, q) with $q \notin \mu(b)$.*

If this condition is not satisfied for some pair (b, q) , we say that the pair blocks the outcome $(u, w; \mu)$.

However, the modern cooperative equilibrium analysis of two-sided matching models relies on a stronger notion than core. The appropriate concept, known as *setwise-stability* (*stability* for short), is the continuous version of the stability concept introduced in Sotomayor [23], which in its turn is an extension of the group-stability concept of Roth [18]. Setwise-stability captures the idea that instabilities may be caused by coalitions of any size. That is, *a feasible payoff x is **setwise-stable** if there is no coalition of players who by forming **new** partnerships only among themselves, possibly dissolving some partnerships of x to remain within their quotas and possibly keeping other ones, can all obtain a higher payoff than the one given by x .*

The coalition in this definition is not required to be able to form all their partnerships only among themselves. This makes the difference between the concepts of setwise-stability and corewise-stability and implies that setwise-stable payoffs are in the core. In the next section it will be illustrated a situation in which a core payoff is not setwise-stable.

The solution concept for market M^* will be called competitive equilibrium outcome. A **competitive equilibrium outcome** is a vector of prices for the objects plus a feasible allocation and a corresponding array of payoffs for the buyers, such that each buyer is assigned to an allowable set of objects in her demand set and every unsold object has a zero price.

These two concepts are put together in Definition 4 below.

Definition 4. a) *The outcome for M , $(u, w; \mu)$, is **setwise-stable (or stable)** if it is feasible and for all $b \in B$ and $S \subseteq Q$, with $|S| \leq r(b)$, we have $U_b \geq \sum_{q \in S} (v_{bq} - w'_{bq})$, where $w'_{bq} = w_q(\min)$ if $q \notin \mu(b)$ and $w'_{bq} = w_{bq}$ if $q \in \mu(b)$. If $(u, w; \mu)$ is a stable outcome we say that (u, w) is a stable payoff.⁸*

b) *The outcome (u, p, μ^*) is a **competitive equilibrium outcome** for M^* if (i) it is feasible, (ii) μ^* is a feasible allocation such that, if $\mu^*(b) = S$ then $S \in D_b(p)$ for all $b \in B$ and (iii) $p_q = 0$ if object q is left unsold.*

If $(u, p; \mu^)$ is a competitive equilibrium outcome we say that (u, p) is a competitive equilibrium payoff, (p, μ^*) is a competitive equilibrium and p is a competitive equilibrium price or an **equilibrium price** for short.*

If there is an allocation μ^* , compatible with p and satisfying condition (ii) of Definition 4-b, we say that p is a **competitive price vector**. The allocation μ^* is said to be **compatible** with the competitive price p . The allocation μ^* is called **competitive** if it is compatible with a competitive price.

For some markets, instabilities can be restricted to pairs of agents of opposite sides and then setwise-stability is equivalent to pairwise-stability. For some other markets the setwise-stability concept is given by corewise-stability. Sotomayor [24] proved that setwise-stability is a new cooperative equilibrium concept, stronger than pairwise-stability plus corewise-stability. The main point however is that **any stable outcome must be in the core**. For the present model, **corewise-stability is not equivalent to setwise-stability**. This fact will be proved in the next section.

It is not hard to see that Definition 4-a is equivalent to require that for all pairs b and q , such that (b,q) is not a partnership, the sum of any b 's individual payoff with any q 's individual payoff is not less than v_{bq} . Therefore, Definition 4-a is equivalent to Definition 3. This means that **an outcome is stable for M if and only if it is pairwise-stable**.

In order to compare two outcomes, agents compare their corresponding total payoffs. Thus, we can define:

Definition 5. *A stable (respectively, competitive equilibrium) payoff is called a B-optimal stable (respectively, competitive equilibrium) payoff if every player in B weakly prefers it to any other stable (respectively, competitive equilibrium) payoff. That is, the B-optimal stable (respectively, competitive equilibrium) payoff gives to each player in B the maximum total payoff among all stable (respectively, competitive equilibrium) payoffs. Similarly we define a Q-optimal stable (respectively, competitive equilibrium) payoff.*

⁸ By the symmetry of the model, the roles between B -players and Q -players can be reversed in Definition 4-a.

The existence and uniqueness of the B -optimal and of the Q -optimal stable payoffs are proved in Sotomayor [23]. The existence and uniqueness of the optimal competitive equilibrium payoffs for each side of the market follow from Theorem 1 presented in section 4.

2.1. COMPETITIVE BEHAVIOR

Unlike his cooperative behavior, **every seller sells all of his items for the same price.** In fact, if a seller has two identical objects, q and q' , and $p_q > p_{q'}$ for some price vector p , then no buyer b will demand, at prices p , a set S of objects that contains object q . This is because, by replacing q with q' in S , b gets a more preferable allowable set of objects. But then, q will remain unsold with a positive price, which violates condition (iii) of Definition 4-b.

It must be pointed out that this sort of event does not have anything to do with the quotas of the buyers. It is due to the assumption of the model under which no buyer is interested in acquiring more than one item of a given seller. Without this restriction, if for example a seller sells all of his objects to the same buyer at competitive equilibrium prices, these prices can be distinct.

Remark 1. When every seller sells his identical objects for the same feasible price, we can identify a seller with any of his objects. Thus, we do not cause any confusion by using the same notation for a seller and for any of his objects. Under this observation, if μ^* is a feasible allocation and μ is its corresponding matching, $q \in \mu^*(b)$ means that object q

is allocated to buyer b (there is only one object q allocated to buyer b), and $q \in \mu(b)$ means that buyer b and seller q are partners at μ .

On the other hand, since the array of payoffs for any seller q is given by the array of prices of his objects, then, in order to represent the array of the $s(q)$ identical individual payoffs for any seller q , we do not need to make any reference to the buyers who are matched to q . For example, (p_q, p_q, \dots, p_q) denotes the array of payoffs of seller q and p_q denotes the price of any of his objects. ■

According to the exposed above, given an equilibrium price, if a seller does not sell all of his objects at these prices, then any of his objects has zero price. **Therefore, at equilibrium prices, every seller with a positive price will sell all of his items and the number of objects in the market will be enough to meet the demand of all buyers.** However, if the identical objects are owned by different sellers, they need not to be sold at the same price. See example below.

Example 2.1.1. (Identical objects owned by different sellers may have different competitive prices) Let $B = \{1, 2, 3, 0\}$, $Q = \{j, k, q, 0\}$, $r(1) = r(2) = r(3) = 2$, $s(j) = 2$, $s(k) = 3$, $s(q) = 1$. The values $v_i = (v_{ij}, v_{ik}, v_{iq}, v_{i0})$, with $i \in B$, are given by: $v_1 = (6, 6, 1, 0)$, $v_2 = (4, 4, 2, 0)$ and $v_3 = (3, 3, 1, 0)$. It is a matter of verification that the B -optimal competitive equilibrium payoff allocates the objects of j to 1 and 2 at price $p_j = 2$; the three objects of k to the three buyers at price $p_k = 0$ and the object of q to buyer 3 at price $p_q = 0$. Hence, the prices of the objects of sellers j and k are not the same although all of them have the same value to any buyer. ■

Proposition 1, whose proof is presented in the Appendix, implies that the prices in the example above would not be distinct if the two sellers had the same quotas.

Proposition 1. *Let (u,p) be the B -optimal competitive equilibrium payoff for M^* . Let $j, k \in Q$, such that $s(j)=s(k)$ and $v_{bj}=v_{bk}$ for every $b \in B$. Then, $p_j=p_k$.*

3. EXAMPLE

In this section we present an example to illustrate the main results of this paper, which can be synthesized as follows: *The set of competitive equilibrium payoffs is a complete lattice that reflects the same polarization of interests between buyers and sellers that characterizes the complete lattice of the set of stable payoffs. This set is a subset and may be a proper subset of the set of stable payoffs, which in its turn may be a proper subset of the core. Moreover, unlike the one-to-one case, the core is not a lattice and the polarization of interests observed in the other two sets does not carry over the core payoffs: the best core payoff for the buyers is not necessarily the worst core payoff for the sellers.*

Then consider the following situation. The B -players will be called firms and the Q -players will be called workers. There are two firms, b and b' , and two workers q and q' . Each firm may employ and wants to employ both workers; worker q may take, at most, one job and worker q' may work and wants to work for both firms. The first row of matrix v is (3,2) and the second one is (3,3).

There are two optimal matchings: μ and μ' , where $\mu(b)=\{q,q'\}$, $\mu(b')=\{q', 0\}$ and $\mu'(b')=\{q,q'\}$, $\mu'(b)=\{q', 0\}$. The core is described by the set of individual payoffs whose

total payoffs satisfy the following system of inequalities: $0 \leq U_b \leq 2$, $0 \leq U_{b'} \leq 3$; $W_q + W_{q'} \geq 3$, $W_{q'} - W_q \leq 2$, $1 \leq W_q \leq 3$. It is not hard to see that the outcome (u, w) is stable if and only if seller q always gets payoff $w_q = 3$, seller q' gets individual payoffs $w_{bq'} \in [0, 2]$ and $w_{b'q'} \in [0, 3]$; the individual payoffs of buyers b and b' are given by $(u_{bq} = 0, u_{bq'} = 2 - w_{bq'})$ and $(u_{b'q'} = 3 - w_{b'q'}, u_{b'0} = 0)$, respectively.

To see that corewise-stability is not adequate to define the cooperative equilibrium for this market, let $(u, w; \mu)$ be such that $u_{bq} = 1$, $u_{bq'} = 1$, $u_{b'q'} = 1$, $u_{b'0} = 0$; $w_{bq} = 2$, $w_{bq'} = 1$, $w_{b'q'} = 2$. That is, firm b hires workers q and q' , obtains from each one of them a profit of 1 and pays 2 to q and 1 to q' ; firm b' hires worker q' at salary 2 and obtains the profit 1. Observe that b' has quota of 2, so it has one unfilled position. It happens that b' can pay more than 2 to q . Thus, if agents can communicate with each other and behave cooperatively, this outcome will not occur, because worker q will not accept to receive only 2 from firm b , since she knows that she can get more than 2 by working with firm b' . Hence, this outcome cannot be a cooperative equilibrium. Observe that $2 = u_{b'0} + w_{bq} < v_{b'q} = 3$, so this outcome is not stable. On the other hand, it is in the core. In fact, if there is a blocking coalition then it must contain $\{b', q\}$. These agents cannot increase their total payoffs by themselves; b' needs to hire both workers. However $\{b', q, q'\}$ does not block the outcome, because q' gets worse by taking only one job. Nevertheless, the coalition of all agents do not block the outcome, since b loses worker q , so it will be worse off.

Now, consider the outcome $(u', w'; \mu)$, where $u'_{bq} = 0$, $u'_{bq'} = 1$, $u'_{b'q'} = 1$, $u'_{b'0} = 0$; $w'_{bq} = 3$, $w'_{bq'} = 1$, $w'_{b'q'} = 2$. Firm b' cannot offer more than 3 to worker q , so the structure of the outcome cannot be ruptured. Then, although both outcomes, $(u, w; \mu)$ and $(u', w'; \mu)$, are corewise-stable, only the second one can be expected to occur, so only this

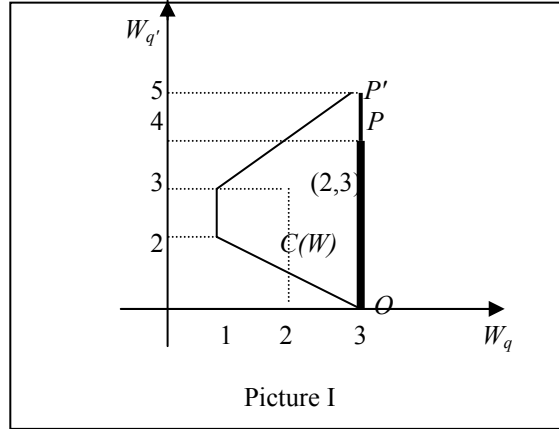
outcome is a cooperative equilibrium. Our explanation for this is that only $(u', w'; \mu)$ is stable.

The connection between the core, the set of stable payoffs and the set of competitive payoffs, exhibited in this example, can be better understood via picture 1 below. Now, if the reader prefers, sets B and Q are better interpreted as being the set of buyers and the set of sellers, respectively.

In picture 1, $C(W)$ is the set of seller's total payoffs, which can be derived from any core payoff. The segment OP' is the set of sellers total payoffs which can be derived from any stable payoff. That is, $(W_q, W_{q'}) \in OP'$ if and only if there is a stable outcome $(u, w; \mu)$ such that $W_q = w_{bq}$ and $W_{q'} = w_{bq'} + w_{b'q'}$. The segment OP is the set of seller's total payoffs, which can be derived from any competitive equilibrium price. That is, $(W_q, W_{q'}) \in OP$ if and only if there is a competitive equilibrium price p such that $W_q = p_q$ and $W_{q'} = p_{q'} + p_{q'}$.

We can see that $C(W)$ is bigger than OP' which, in its turn, is bigger than OP . The point $(2, 3) \in C(W) - OP'$. It corresponds to the outcome (u, w, μ) described above that is in the core but is not stable.

It is clear in Picture 1 that $C(W)$ is not a lattice, so the core is not a lattice under neither \geq_B nor \geq_Q . In fact, the outcome which corresponds to the point $(3, 0)$ gives the individual payoff of 3 to q and two individual payoffs of 0 to q' . On the other hand, the core outcome that corresponds to $(2, 3)$ gives the individual payoff of 2 to q . Then, the infimum (respectively supremum) of these two core payoffs under \geq_Q (respectively \geq_B) gives payoff 2 to seller q and two individual payoffs of 0 to seller q' . This payoff corresponds to the vector of total payoffs $(2, 0)$, not in $C(W)$.



It is evident that the stable payoff corresponding to point $P'=(3,5)$ is the Q -optimal stable payoff. Seller q receives 3 and seller q' receives 2 from b and 3 from b' . Buyers get 0 from the sellers. By applying the function f we obtain the Q -optimal competitive equilibrium payoff, corresponding to point $P=(3,4)$, where q receives 3 and q' receives 2 from both buyers.

Point $O=(3,0)$ corresponds to the outcome $(u'',w'';\mu)$, where $u''_{bq}=0$, $u''_{bq'}=2$, $u''_{b'q}=3$, $u''_{b'o}=0$; $w''_{bq}=3$, $w''_{bq'}=0$, $w''_{b'q}=0$. Payoff (u'',w'') is the B -optimal stable payoff, the B -optimal competitive equilibrium payoff and the B -optimal core payoff. It is the worst stable payoff and the worst competitive equilibrium payoff for the sellers. However, it is not the worst core payoff for the sellers since it is the best core payoff for q . Indeed, as it can be observed, there is no minimum core payoff for the sellers.

4. MAIN CONNECTION BETWEEN THE STABLE AND THE COMPETITIVE EQUILIBRIUM PAYOFFS. THE LATTICE PROPERTY.

To better understand the results presented in this section, the following discussion is in order. The definition of a partial order relation in the set of stable payoffs seems to be

complicated, due to the fact that the array of individual payoffs is unordered sets of numbers. The agents' preferences do not define a partial order relation in this case, since they violate the anti-symmetric property. This problem was solved in Sotomayor [23]. According to Theorem 1 of this paper we can represent the array of individual payoffs of a player as a vector in a Euclidean space, whose dimension is the quota of the given player. This representation is independent of the matching. For example, if player b has quota 5 and forms partnerships with q_1, q_2, q_3, q_4 and q_5 under some stable outcome $(u, w; \mu)$, then the array of payoffs of b is given by $\{u_{b1}, u_{b2}, u_{b3}, u_{b4}, u_{b5}\}$. Now, suppose we have more than one optimal matching for M , and only the partnerships with q_1 and q_2 are maintained by b in all the optimal matchings. The theorem mentioned above proves that $u_{b3} = u_{b4} = u_{b5} = u_b(\min) \equiv \lambda$. Thus, we can choose the following ordering for the 5-tuples of u_{bq} 's: $(u_{b1}, u_{b2}, \lambda, \lambda, \lambda)$. Under any other stable outcome $(u', w'; \mu')$ the vector of individual payoffs of player b can be represented by $(u'_{b1}, u'_{b2}, \lambda', \lambda', \lambda')$, with $\lambda' \equiv u'_b(\min)$. We compare the two vectors of individual payoffs by using the natural partial order relation of R^5 .

Therefore, by ordering the players in B (respectively, Q), we can immerse the stable payoffs of these players in a Euclidean space, whose dimension is the sum of the quotas of all players in B (respectively, Q). Then, the natural partial order relation of this Euclidean space induces the partial order relation \geq_B (respectively \geq_Q) in the set of stable payoffs. We say that $(u, w) \geq_B (u', w')$ **if the vector of individual payoffs of any buyer, under (u, w) , is greater than or equal to her vector of individual payoffs under (u', w') . Similarly we define $(u, w) \geq_Q (u', w')$.**

The vectorial representation of the stable payoffs allowed the following properties to be proved in Sotomayor [23]. They will be used in the proofs of the results of this section:

P1. *A matching is compatible with a stable payoff if and only if it is optimal.*

One of the implications of P1 is that, if a seller has some unsold object under a stable outcome, then one of his individual payoffs will be zero under any other such outcome.

An important feature of the set of stable payoffs is the “polarization of ordering” that exists between the two sides of the market along the whole set of such payoffs. That is:

P2. *If (u,w) and (u',w') are stable payoffs then $(u,w) \geq_B (u',w')$ if and only if $(u',w') \geq_Q (u,w)$.*

An implication of Property P2 is **the conflict of interests that exists between the two sides of the market with respect to two comparable stable payoffs**. That is, if payoffs (u,w) and (u',w') are stable and comparable, then $U_b \geq U'_b$ for all b if and only if $W'_q \geq W_q$ for all q . The main characterization of the set of stable payoffs is given by P3.

P3. *The set of stable payoffs is a convex and complete lattice under the partial orders \geq_B and \geq_Q .*⁹

The lattice property implies that there exist a maximal element and a minimal element in the set of stable payoffs. The fact that the lattice is complete implies the uniqueness of these extreme points. Thus, there exist one and only one stable payoff (u^*,w^*) and one and only one stable payoff (u_*, w_*) such that $(u^*,w^*) \geq_B (u',w')$ and

⁹ A set L , endowed with a partial order relation, has the lattice property if $\sup\{x, x'\}$ and $\inf\{x, x'\}$ are in L , for all $x, x' \in L$. The lattice is complete if every subset has a supremum and an infimum. (See Birkhoff [3]).

$(u^*, w^*) \geq_Q (u', w')$ for all stable payoffs (u', w') . This lattice property is of interest because these two extreme points have important meaning for the model. Even though the preferences of the players do not define the partial orders \geq_B and \geq_Q , we have that $U^*_b \geq U'_b$ and $W^*_q \geq W'_q$ for all $(b, q) \in B \times Q$ and all stable payoffs (u', w') . Then, (u^*, w^*) (respectively (u_*, w_*)) is the B -optimal (respectively Q -optimal) stable payoff as defined in Definition 5. Due to the polarization of interests pointed above, the "best" stable payoff for a given side is the "worst" stable payoff for the other side. Formally, **any player in Q (respectively, B) weakly prefers any stable payoff to the B -optimal stable payoff (respectively, Q -optimal stable payoff).**

Let E denote the set of stable payoffs for market M . The main result is given by Theorem 1.

Theorem 1. *The set of competitive equilibrium payoffs is a non-empty and complete lattice under the partial order \geq_B (respectively \geq_Q) whose supremum (respectively infimum) is B -optimal and whose infimum (respectively supremum) is Q -optimal.*

The plan of the proof is the following. First, Lemma 1 **characterizes the competitive equilibrium payoffs as the stable payoffs in which sellers do not discriminate the buyers.** Then, Lemma 2 suggests to us that we can obtain the set of competitive equilibrium payoffs by "shrinking" the set E through the application of a convenient map f from E to E . A straightforward application of Lemma 1 determines that **the fixed points of f are exactly the competitive equilibrium payoffs.** We then prove that such a function is *isotone* (order-preserving). The desired result is concluded via

an *algebraic* fixed point theorem due to Alfred Tarski [29]¹⁰, by capitalizing on the lattice structure of the set of stable payoffs:

Tarski's Theorem. *Let E be a complete lattice with respect to some partial order \geq , and let f be an isotone function from E to E . Then the set of fixed points of f is non-empty and is itself a complete lattice with respect to the partial order \geq .*

The proofs of Lemma 1 and Lemma 2 are given in the Appendix.

Lemma 1. *a) If (u, p, μ^*) is a competitive equilibrium outcome for M^* then (u, p, μ) is a stable outcome for M , where μ corresponds to μ^* . b) If (u, p, μ) is a stable outcome for M and the array of payoffs for any seller q is given by $s(q)$ repetitions of the number p_q , then (u, p, μ^*) is a competitive equilibrium outcome for M^* , where μ^* corresponds to μ .*

It follows from Lemma 1 and property P1 that, if (u, p, μ^*) is a competitive equilibrium outcome then μ^* is an optimal allocation; if μ^* is an optimal allocation then it is compatible with any competitive equilibrium outcome. Some interesting implication of this is that, if a seller has some unsold object under a competitive equilibrium, then he will get total payoff of 0 under any other such outcome.

¹⁰ Roth and Sotomayor [19] is the first paper to use this theorem in Matching Theory. In this paper, a set of interior points of the core consists of the fixed points of a "rebargaining" function. Tarski's theorem itself has been used to study the equilibria of non-cooperative games in Topkis [31] and Vives [32], and algebraic fixed

Lemma 2. Let $(u, w; \mu)$ be a stable outcome. Set $w'_{bq} \equiv w_q(\min)$, if $b \in \mu(q)$. Let u' be the respective payoff for the buyers. Then, $u'_{bq} \geq u_{bq}$, for all $(b, q) \in B \times Q$ with $b \in \mu(q)$ and $(u', w'; \mu)$ is stable.

Proof of Theorem 1. Take an optimal matching μ for M . If $(u, w) \in E$, then construct the payoff (u', w') such that the individual payoffs of a seller q are given by a vector with $s(q)$ repetitions of the number $w'_q = w_q(\min)$, and the u'_{bq} 's are given by $u'_{bq} = v_{bq} - w'_q$ if $q \in \mu(b)$. Define $f(u, w) \equiv (u', w')$. The result will follow immediately from Tarski's Theorem once we establish that

- (1) $f: E \rightarrow E$
- (2) $f(x) = x$ if and only if x is a competitive equilibrium payoff.
- (3) f is an isotone function (i.e., f preserves the partial orders \geq_B and \geq_Q).

For statement (1), let $(u, w) \in E$ and let $f(u, w) = (u', w')$ as defined above. We want to show that $(u', w') \in E$. First observe that Property P1 implies that $(u, w; \mu)$ is a stable outcome for M . Now, use Lemma 2.

For statement (2), if $f(x) = x$ the conclusion follows from Lemma 1-b; conversely, if x is a competitive equilibrium payoff then Lemma 1-a implies that $x \in E$. The conclusion that $f(x) = x$ is then immediate from the definition of f and the fact that under x , the individual payoffs of any seller q are given by $s(q)$ repetitions of the same number.

We will prove (3) by showing that f preserves \geq_B . The other case follows dually. Then, take (u^1, w^1) and (u^2, w^2) in E , with $(u^1, w^1) \geq_B (u^2, w^2)$. Set $f(u^j, w^j) \equiv (u^{*j}, w^{*j})$ for

points theorems closely related to Tarski's theorem (Roth [16] and Blair and Roth [2]) were used in

$j=1,2$. It follows from property P2 that $w^1 \leq w^2$, so $w^1_{bq} \leq w^2_{bq}$, for all $q \in \mu(b)$, so $w^1_{q(\min)} \leq w^2_{q(\min)}$, for all $q \in Q$ and so $w^{*1}_q \leq w^{*2}_q$ for all $q \in Q$. Therefore, $w^{*1} \leq w^{*2}$. On the other hand, $(u^{*j}, w^{*j}) \in E$ for all $j=1,2$ and then P2 implies that $u^{*1} \geq u^{*2}$. Hence, $f(u^1, w^1) \geq_B f(u^2, w^2)$ and we have completed the proof. ■

We must point out that $f(x)$ is a **competitive equilibrium payoff for all $x \in E$** . This fact follows from Lemma 1-b applied to $f(x)$. Therefore, **the image of the set of stable payoffs under f is the set of competitive equilibrium payoffs**.

The fact that the set of competitive equilibrium payoffs is a subset of E implies that the competitive equilibrium payoffs reflect the same kind of polarization of ordering and interests that characterizes the set of stable payoffs. In addition, **buyers have interests opposed to those of the sellers, along the whole set of the competitive equilibrium payoffs** (the converse may not be true).

The lattice property of the set of competitive equilibrium payoffs under \geq_Q , plus the fact that all of the items owned by a seller have the same price imply that the set of competitive equilibrium prices is itself a complete lattice, under the partial order relation defined by the preferences of the sellers. The *minimum competitive equilibrium price* corresponds to the B-optimal competitive equilibrium payoff and the *maximum competitive equilibrium price* corresponds to the Q-optimal competitive equilibrium payoff.

Another important result is given by Theorem 2, which asserts that the isotone function f maps the two extreme points of the lattice of the stable payoffs into the two respective extreme points of the lattice of the competitive equilibrium payoffs.

cooperative game theory in Roth [17].

Theorem 2. *Let (u, w) be the Q -optimal (respectively, B -optimal) stable payoff for the related cooperative market M . Then, $f(u, w)$ is the Q -optimal (respectively, B -optimal) competitive equilibrium payoff.*

Proof. Suppose (u, w) is the Q -optimal stable payoff. Set $f(u, w) = (u^*, w^*)$. We have already proved that $(u^*, w^*) \in E$. Lemma 1-b plus the definition of f imply that (u^*, w^*) is a competitive equilibrium payoff. If (u^*, w^*) was not the Q -optimal competitive equilibrium payoff then there would be some competitive equilibrium payoff (u', w') such that $w'_q > w^*_q$ for some q . However, Lemma 1-a implies that (u', w') is stable for M , so $w'_q = w'_q(\min) \leq w_q(\min)$ by the Q -optimality of (u, w) . But then $w'_q \leq w^*_q$, which is a contradiction. Thus, we have proved that $f(u, w)$ is the Q -optimal competitive equilibrium payoff.

Suppose now that (u, w) is the B -optimal stable payoff. A consequence of Lemma 2 and of the Q -minimality of (u, w) is that no seller discriminates his partners under this outcome. This means that, for any optimal matching μ , we must have that $w_{bq} = w_q(\min)$ for all $(b, q) \in B \times Q$ with $b \in \mu(q)$. Then, it follows from Lemma 1-b that $(u, w; \mu)$ is a competitive equilibrium payoff. Therefore, $f(u, w) = (u, w)$. We want to show that (u, w) is the B -optimal competitive equilibrium payoff. Let (u', w') be the B -optimal competitive equilibrium payoff. By Lemma 1-a, $(u', w') \in E$ and then

$$u \geq u' \quad (1)$$

by the maximality of (u, w) in E . That (u, w) is a competitive equilibrium payoff follows from the fact that it is a fixed point of f (and also follows from Lemma 1-b). Then,

$$u \leq u' \quad (2)$$

by the maximality of (u', w') . Then, $u = u'$ by (1) and (2). Since both stable payoffs are compatible with the same optimal matching μ , by property P1, this implies that $w = w'$. Hence, (u, w) is the B -optimal competitive equilibrium payoff and the proof is complete. ■

It follows from the second part of the proof of Theorem 2 that the B -optimal stable payoff is a fixed point of f . This theorem then implies that such outcome is the B -optimal competitive equilibrium payoff. Formally,

Corollary 1. *Let (u, w) be the B -optimal stable payoff for M . Let μ be an optimal matching. Then, (i) $w_{bq} = w_{b'q}$ for all $q \in Q$ and all b and b' in $\mu(q)$. (b and b' might be dummy players) and (ii) (u, w) is the B -optimal competitive equilibrium payoff.*

Symmetrical conclusion to part (i) can be obtained by reverting the roles between B -players and Q -players.

It must be pointed out that the function f is not the identical map. In the situation described in the example of section 3, the Q -optimal stable payoff for M is not a fixed point of f , so it is not a competitive equilibrium payoff.

5. THE ROLE OF QUOTAS, UTILITIES, MULTIPLICITY OF PARTNERSHIPS AND RULES OF THE GAME.

The quotas, utilities, multiplicity of partnerships and rules of the game are the main characteristics of the two-sided matching models. While in our model agents have quotas,

multiple partners, additively separable utilities, and an outcome specifies the individual payoffs of each player, the model presented in Kelso and Crawford [13], for example, has no quotas, the multiplicity of partnerships is allowed to only one of the sides of the market, the utilities satisfy the gross-substitute condition and are monotone,¹¹ but are not necessarily additively separable, and an outcome only specifies the total payoff of the players.

This section discusses how these intrinsic distinctions between these two models reflect in their results. Unlike we have proved here, the appropriate cooperative equilibrium concept for the model of Kelso and Crawford is the core concept; the core coincides with the set of competitive equilibrium payoffs. As to the lattice property, a less general result than ours is proved by Gul and Stacchetti [10] the set of competitive equilibrium prices that are compatible to the same matching is endowed with the lattice property under the partial order defined by the preferences of the sellers.

The first finding leads us to conclude that the results for the model of Kelso and Crawford are not affected by the fact that buyers have no quota. Indeed, we can introduce quotas in that model without causing any modification in those results. To better understand the procedure, the following preliminary discussion is in order. Quotas play a fundamental role in modeling a two-sided matching game. They determine the maximum size of the allowable sets of partners to each player. Players consider as unacceptable any set of partners that is not allowable to them. "No quota" is a particular and trivial case. It

¹¹ The demand set of buyer b at prices p is defined as $D_b(p) = \{S \subseteq Q; P(\{b\} \cup S) - p_S \geq P(\{b\} \cup T) - p_T, \text{ for all } T \subseteq Q\}$, where $p_S = \sum_{i \in S} p_i$. The preferences of a buyer b satisfies the gross substitute condition if for any two price vectors p and p' such that $p' \geq p$, and any $A \in D_b(p)$, there exists $B \in D_b(p')$ such that $\{q \in A; p_q = p'_q\} \subseteq B$.
If $u: 2^Q \rightarrow \mathbb{R}$, with $u(S) = P(\{b\} \cup S)$, denotes the utility to firm b of hiring the set of workers $S \subseteq Q$, the utility function u is monotone if for all $A \subseteq B \subseteq Q$, $u(A) \leq u(B)$.

means that every agent may form partnerships with any number of agents of the opposite side, so the quota of each agent is the total number of agents of the opposite side.

Actually, the quota of an agent can be endogenously defined by using the characteristic function $P(\cdot)$. We say that player $y \in B \cup Q$ has quota of $k(y)$, if $k(y)$ is the smaller integer number such that, for all sets S , formed with potential partners of y , with $|S| \geq k(y)$, $P(\{y\} \cup S) = \max \{P(\{y\} \cup S'); S' \subseteq S \text{ and } |S'| \leq k(y)\}$.

When $P(\{y\} \cup \cdot)$ is monotone, it is enough to require that $|S| = k(y)$ in this definition. Thus, if for instance $Q = \{q_1, q_2, q_3\}$ and the function $P(\cdot)$ is such that $P(\{b\} \cup \{q_1\}) = P(\{b\} \cup \{q_2\}) = 3$, $P(\{b\} \cup \{q_3\}) = 2$, $P(\{b\} \cup \{q_1, q_2\}) = 5$, $P(\{b\} \cup \{q_1, q_3\}) = 4$, $P(\{b\} \cup \{q_1, q_2, q_3\}) = 5$, it is easily seen that $r(b) = 2$.

We say that allocation μ is **feasible** if

$$|\mu(y)| \leq k(y) \text{ for all } y \in B \cup Q. \quad (1)$$

Thus, the quota of a player is the maximum number of partnerships he/she/it can form under a feasible matching.

Now, observe that the set of competitive equilibrium prices, and consequently the core payoffs and the results in question for the model of Kelso and Crawford do not change if quotas are introduced in that model. This is due to the fact that **p is a competitive equilibrium price** (respectively, $(u, w; \mu)$ is a core outcome) **for the model without quotas if and only if it is a competitive equilibrium price** (respectively, core outcome) **for the model with quotas**. In fact, if $(p; \mu)$ is a competitive equilibrium (respectively, $(u, w; \mu)$ is a core outcome) for the model without quotas and $|\mu(b)| = r(b) + k$, $k > 0$, then there exist k objects in $\mu(b)$, say q_1, \dots, q_k , with price 0. This is because the utilities are monotone and $\mu(b)$ is a set of objects demanded by b . Therefore, there is a feasible

allocation μ' , under which q_1, \dots, q_k are left unsold and such that $(p; \mu')$ is a competitive equilibrium (respectively, $(u, w; \mu')$ is a core outcome). Since μ' satisfies (1), $(p; \mu')$ is a competitive equilibrium (respectively, $(u, w; \mu')$ is a core outcome) for the model with quotas.

However, the rules of the game, the agents' utilities and the multiplicity of partnerships may affect the results in question. According to the rules of the game, an outcome specifies the agents' individual payments - **(R1)** - or agents do not care about their individual payments, so an outcome only specifies the agents' total payoffs - **(R2)**. The rules corresponding to R1 are not completely described by the characteristic function of the game. The fact that agents are allowed to negotiate their individual payoffs causes **the set of setwise-stable payoffs to be equal to the set of pairwise-stable payoffs and, some times, to be different from the core. Hence, the set of pairwise-stable payoffs is a subset of the core and may be smaller than this set.** This is the case of our model. Since buyers are not allowed to buy more than one object from the same seller and no seller can discriminate the buyers under a competitive equilibrium payoff, then **the set of competitive equilibrium payoffs may be smaller than the set of pairwise-stable payoffs.**

The rules corresponding to R2 are completely specified by the characteristic function of the game. Agents act in block, so they only care about their total payoffs. Consequently, **the cooperative equilibrium is given by corewise-stability and, some times, it is different from pairwise-stability. Hence, the set of pairwise-stable payoffs may be bigger than the core.** This is the case of the many-to-one matching model of Kelso and Crawford (**the substitutability of the preferences causes buyers to act in blocks**) and of one of the many-to-many matching models with additively separable

utilities treated in Sotomayor [22]. In the former model, as a seller only owns one object then he cannot discriminate buyers, so **the core coincides with the set of competitive equilibrium payoffs in this model**. Since prices are not specified in the later model, there is no sense of talking about competitive prices (prices are seller's individual payoffs).

A particular case of the model of Kelso and Crawford with quotas is the many-to-one matching model presented in Sotomayor [27], where utilities are additively separable and rules correspond to R2. If we change the rules from R2 to R1, we obtain the many-to-one version of the model presented here. The additiveness of the preferences and the fact that sellers have quota of one imply that a buyer is always able to negotiate her individual payoffs, even under R2. This makes the set of pairwise-stable payoffs, the core and the set of competitive equilibrium payoffs to coincide.

The lattice property of the set of sellers' stable payoffs is affected by the rules R1 and R2 in the many-to-many case. In fact, under additively separable utilities, the set of sellers' stable payoffs forms a lattice under \geq_Q , when the rules of the game correspond to R1, as proved in section 4. This set may not to be a lattice when rules are changed to R2, as illustrated by the set of sellers' core payoffs of the example given in section 3.¹² Consequently, this set is not a lattice when utilities satisfy the gross-substitute condition and are monotone and the multiplicity of partners is allowed to both sides.

For the many-to-one matching model, the lattice property of the set of sellers' stable payoffs is not affected by the rules R1 and R2 and seems to be related to the gross-substitute condition. In fact, if utilities are additively separable, the payoff of a buyer under R2 is the sum of her corresponding individual payoffs under R1. Hence, **the set of buyers'**

¹² It can be shown that the total core payoffs of the model under R1 are the core payoffs of the model under R2.

stable payoffs under R2 is the set of their total payoffs, derived from their corresponding individual stable payoffs under R1. Consequently, **the set of sellers' stable payoffs under R1 equals the set of sellers' stable payoffs under R2.** This implies that the lattice property of the set of sellers' stable payoffs is not affected by the modification of the rules in the many-to-one case. The set of sellers' stable payoffs forms a lattice under R1 and under partial order \geq_Q , because it is a lattice for the corresponding many-to-many case. Then, the set of sellers' stable payoffs forms a lattice under R2 and under partial order \geq_Q . Since sellers have quota of one, this partial order coincides with the one defined by the preferences of the sellers (the relation \geq_B is not defined under R2).

When utilities are not additively separable but satisfy the gross-substitute condition and are monotone, we still have the lattice property of the set of sellers' stable payoffs which are compatible to the same matching. Otherwise, if buyers' utilities do not satisfy gross-substitute condition, the lattice property of the set of competitive equilibrium prices may fail to hold. See the following example.

Example 5.2. The set of buyers is $B=\{b, b'\}$ and the set of sellers (objects) is $Q=\{q_1, q_2, q_3\}$. The utility functions of the players are given by the characteristic function $P(\cdot)$, with $P(\emptyset)=P(S)=0$ if $S\subseteq B$ or $S\subseteq Q$. The other values $P(S)$ are given in the matrix below.

	q_1	q_2	q_3	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_2, q_3\}$	$\{q_1, q_2, q_3\}$
b	3	3	2	5	4	4	5
b'	2	1	2	2	2	2	3
$\{b, b'\}$	3	3	2	6	5	5	7

Table I

The entry 1, for example, is the value $P(\{b'\} \cup \{q_2\})$. Observe that the quotas of the players are: $r(b)=2$, $r(b')=3$, $s(q_1)=s(q_2)=s(q_3)=1$.

Now, consider the feasible allocation μ where $\mu(b)=\{q_1, q_2\}$ and $\mu(b')=\{q_3\}$. It is easy to see that the outcomes $(u, p; \mu)$ and $(u', p'; \mu)$ are in the core, where $p=(1, 0, 1)$, $u_1=4$, $u_2=1$; $p'=(0, 1, 0)$, $u'_1=4$ and $u'_2=2$. Clearly, (p, μ) and (p', μ) are competitive equilibrium.

However, the infimum of $(p; \mu)$ and $(p'; \mu)$ is not a competitive equilibrium under the partial order defined by the preferences of the sellers. In fact, $p \wedge_Q p' = p''$, with $p''=(0, 0, 0)$. This price vector is not a competitive equilibrium price, since there is no feasible allocation that supports p'' . In fact, μ is the only efficient allocation. Under μ , b' gets only q_3 but demands $\{q_1, q_2, q_3\}$. Consequently, there is no core allocation that gives to the sellers the payoff vector p'' . (Observe that $(u'', p''; \mu)$, where $u''_b=5$ and $u''_{b'}=2$ is pairwise stable although it is not in the core). In addition, the fact that p'' is not a competitive equilibrium price implies that there is no minimum equilibrium price.

In this example the utilities of the buyers are monotone but do not satisfy gross substitute condition. In fact, $P(\{b'\} \cup \{q_1, q_2\}) - P(\{b'\} \cup \{q_1\}) = 0 < 1 = P(\{b'\} \cup \{q_1, q_2, q_3\}) - P(\{b'\} \cup \{q_1, q_3\})$. This means that adding q_2 to the set $\{q_1, q_3\}$ increases b_2 's utility more than adding it to set $\{q_1\}$, which violates the decreasing returns (submodularity) of $P(\{b'\} \cup \mathcal{I}_i)$. It is known that gross substitute implies submodularity, so buyer b_2 cannot satisfy gross substitute condition. ■

6. CONCLUDING REMARKS AND RELATED WORK

Along the last two decades, the two-sided matching models have moved from being an interesting set of pure mathematical models to being an important part of the emerging field of market design. Through these models a variety of markets has been able to be better understood, what has considerably helped the institutions that have evolved to organize such markets.

Without seeking for wider generalizations, the present paper contributes with one step further to the understanding of what markets do and what they are. It examines the link between the cooperative and competitive equilibria of a similar environment, modeled as a two-sided many-to-many matching game with separable and additive utilities. Thus, our model is the simplest generalization of the Assignment Game of Shapley and Shubik [21] to the many-to-many case.

The cooperative structure fits well into a variety of labor markets where players are firms and workers. It is important to point out that our game is a new kind of coalitional game with transferable utility payoff. In the traditional coalitional games, the rules of the game are completely described by the coalitional function of the game, an outcome only specifies a number for each player and the concept of equilibrium is given by the core. The novelty here is that the main characteristic of the game: *A pair of players can form a partnership and negotiate their individual payments the way they like*, cannot be described by the coalitional function of the game and an outcome specifies a set of numbers for each player. This causes the stability concept to be stronger and economically more natural than corewise-stability. The set of stable payoffs is contained in the core and may be a proper subset of it, while coincides with the set of pairwise-stable payoffs.

The competitive approach provides an economic structure in which players may be thought of as being buyers and sellers. Buyers are interested in sets of items of different sellers and each seller owns a lot of identical objects. A new equilibrium concept, called competitive equilibrium payoff, is introduced. The paper then examines how this concept, closely related to the traditional concept of equilibrium from standard microeconomic theory, compares with the cooperative equilibrium concept used in this coalitional game.

In Sotomayor [23], the set of stable payoffs is characterized by a complete lattice structure, under a convenient partial order, which reflects a certain polarization of interests between the two sides of the market, with respect to two comparable stable payoffs. The main finding of the present paper is that we can construct a map from the lattice of the stable payoffs into itself, whose fixed points are the competitive equilibrium payoffs. Such a map is order preserving but is not the identity. This having been done, we capitalized on the lattice structure of the set of stable payoffs to draw conclusions about both, the existence and the algebraic structure of the set of competitive equilibrium payoffs. Tarski's theorem then implies that the set of competitive equilibrium payoffs is itself a non-empty complete lattice under both partial orders, \geq_B and \geq_Q . In economic terms, this structure permitted us to make welfare comparisons between different outcomes, e.g., to observe that, although the partial orders are not defined by the preferences of the players, there is one and only one B -optimal (respectively Q -optimal) competitive equilibrium payoff that is simultaneously the best for all buyers and the worst for all sellers. Thus, the set of competitive equilibrium payoffs exhibits the same polarization of interests that characterizes the set of stable payoffs.

Since the lattice structure of the set of stable payoffs appears to be characteristic of many of the two-sided matching markets, the techniques used in this paper can be applied to any such market. What is needed is to construct an order-preserving function from the set of stable payoffs to itself, whose fixed points are those of interest.

Since Shapley and Shubik, the Assignment game has been widely studied and extended to more complex assignment models, by allowing many-to-one and many-to-many matching. Crawford and Knower [4], for example, consider a discrete version (as

well as the continuous version) of the Assignment game and develop a version of the deferred acceptance algorithm of Gale and Shapley [9] to prove the non-emptiness of the set of pairwise-stable payoffs. Kelso and Crawford [13] generalize Crawford and Knoer [4] by considering a many-to-one job market where all workers are gross substitutes from each firm's standpoint. The same market is studied by Gul and Stacchetti [10, 11] in the context of buyers and sellers.

Sotomayor [22] presents two different approaches for the many-to-many matching model with separable and additive utilities. One of them is the cooperative model treated here. In the other one firms and workers negotiate their total payoffs, instead of their individual payments.

In a companion paper we provide a dynamical mechanism that generalizes the one of Demange, Gale and Sotomayor [7] to the many-to-many Assignment game. Both mechanisms find the minimum competitive equilibrium price.

Gul and Stacchetti [11] presents a dynamic mechanism for the many-to-one matching model of Kelso and Crawford [13] that yields the minimum equilibrium price.

Bikhchandani and Mamer [1] analyze the existence of market clearing prices in an exchange economy in which agents have interdependent values over several indivisible objects. Although an agent can be both a buyer and a seller, such exchange economy can be transformed into a many-to-one matching market where each seller owns only one object and buyers want to buy a bundle of objects, and can be viewed as an extension of the Assignment game.

Demange and Gale [6] generalize the Assignment game by allowing that agents' preferences may be represented by any continuous utility functions in the money variable.

Many other works related to the Assignment game can be found in the literature. We can cite Demange [5], Leonard [14], Perez-Castrillo and Sotomayor [15], Sotomayor [25], [26], [27] and [28], Thompson [30], Eriksson and Karlander [8] and Kaneko [12], among others.

APPENDIX

Proof of Proposition 1. Let μ^* be some optimal allocation for M^* . Suppose by way of contradiction that $p_j \neq p_k$. Without loss of generality it can be assumed that $p_j > p_k \geq 0$. Then, seller j must have sold all of his objects at μ^* , and for all b assigned to some object of j we have that $v_{bj} - p_j < v_{bj} - p_k = v_{bk} - p_k$. The competitiveness of p then implies that if b is assigned to some object of j then she is also assigned to some object of k . Since $s(j) = s(k)$ we have that $\mu^*(j) = \mu^*(k)$. We claim that price vector p^* is also a competitive equilibrium price, with matching μ^* , where $p_j^* = p_k^* = p_k$ and $p_q^* = p_q$ for all objects of seller $q \notin \{j, k\}$. We have to show that, if $q \in \mu^*(b)$, then $v_{bq} - p_q^* \geq v_{bq'} - p_{q'}^*$, for every $q' \notin \mu^*(b)$. (This is equivalent to requiring that $\mu^*(b)$ is in $D_b(p^*)$, by Remark 1). There are two cases.

Case 1. $q \in \{j, k\}$. Then, for all $q' \notin \mu^*(b)$, we have that $q' \notin \{j, k\}$ and $v_{bj} - p_j^* = v_{bk} - p_k^* = v_{bk} - p_k \geq v_{bq'} - p_{q'}^* = v_{bq'} - p_{q'}^*$, where the inequality follows from the competitiveness of p . Therefore, $v_{bq} - p_q^* \geq v_{bq'} - p_{q'}^*$.

Case 2. $q \notin \{j, k\}$. If $q' \notin \{j, k\}$, the result is immediate from the competitiveness of p . If $q' \in \{j, k\}$, then $b \notin \mu^*(j) = \mu^*(k)$. We have that,

$$v_{bq} - p_q^* = v_{bq} - p_q \geq v_{bk} - p_k = v_{bj} - p_j^* = v_{bq'} - p_{q'}^*, \text{ if } q' = j, \text{ and}$$

$$v_{bq} - p_q^* = v_{bq} - p_q \geq v_{bk} - p_k = v_{bk} - p_k^* = v_{bq'} - p_{q'}^*, \text{ if } q' = k,$$

where the inequalities follow from the competitiveness of p .

Thus, in any case, $v_{bq} - p_q^* \geq v_{bq'} - p_{q'}^*$. Therefore, p^* is competitive. However, $p^* < p$, which contradicts the minimality of p . Hence, $p_j = p_k$ and the proof is complete. ■

Proof of Lemma 1. If $(u, p; \mu^*)$ is a competitive equilibrium outcome for M^* and μ corresponds to μ^* , then, since $p_q = 0$ for every unsold object q , it follows that $(u, p; \mu)$ is a feasible outcome for M , where the array of payoffs for seller q is given by the $s(q)$ repetitions of p_q . In addition, $\mu^*(b)$ belongs to $D_b(p)$ for every b . Then, for all $b \in B$, $u_{bq} = (v_{bq} - p_q) \geq (v_{bk} - p_k)$ for all $q \in \mu^*(b)$ and $k \notin \mu^*(b)$, from which follows that $u_b(\min) = \min\{u_{bq}; q \in \mu^*(b)\} \geq (v_{bk} - p_k)$ for all $b \in B$ and $k \notin \mu^*(b)$. That is, **$u_b(\min) + p_k \geq v_{bk}$ for all (b, k) with $k \notin \mu(b)$** . Therefore, $(u, p; \mu)$ is a stable outcome for the related cooperative market M and μ is compatible with (u, p) in M . With an analogous argument we can show part b). ■

Proof of Lemma 2. In fact, if $b \in \mu(q)$ for some q , $u'_{bq} = v_{bq} - w'_{bq} = v_{bq} - w_q(\min) \geq v_{bq} - w_{bq} = u_{bq}$. That is, $u'_{bq} \geq u_{bq}$, for all pairs (b, q) with $b \in \mu(q)$, so $u'_b(\min) \geq u_b(\min)$ for all b . Therefore, for every pair (b, q) with $b \notin \mu(q)$, $u'_b(\min) + w'_q(\min) \geq u_b(\min) + w_q(\min) \geq v_{bq}$ by stability of $(u, w; \mu)$, so $(u', w'; \mu)$ does not have any blocking pair. The feasibility of $(u', w'; \mu)$ follows straightforwardly (see that $u'_{bq} \geq u_{bq} \geq 0$ implies $u'_{bq} \geq 0$). Hence, $(u', w'; \mu)$ is stable. ■

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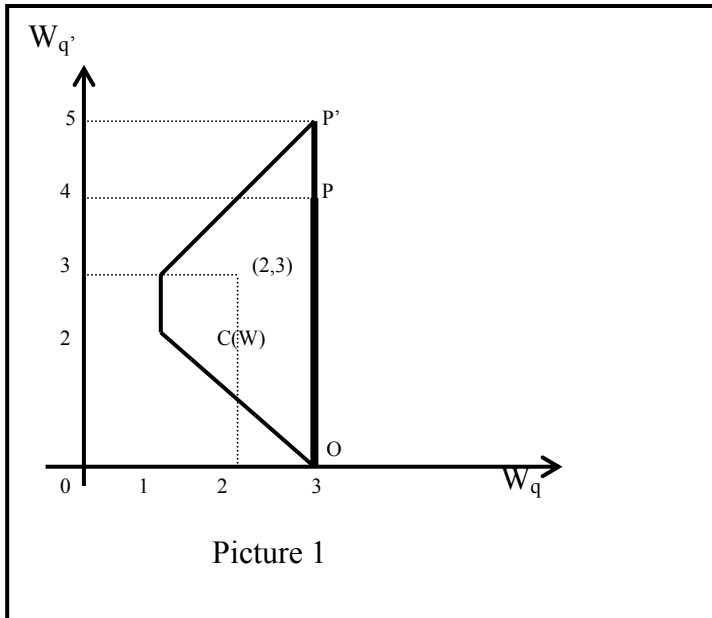
Table for page 34.

	q_1	q_2	q_3	$\{q_1, q_2\}$	$\{q_1, q_3\}$	$\{q_2, q_3\}$	$\{q_1, q_2, q_3\}$
b	3	3	2	5	4	4	5
b'	2	1	2	2	2	2	3
$\{b, b'\}$	3	3	2	6	5	5	7

Table I

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Figure for page 21.



Picture 1