THE STABILITY OF THE EQUILIBRIUM OUTCOMES IN THE ADMISSION GAMES INDUCED BY STABLE MATCHING RULES

by

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ABSTRACT

Suppose a stable matching rule is used as the outcome function for the Admission game introduced in Sotomayor (1996-b) where colleges behave straightforwardly and the students' strategies are given by their preferences over the colleges. We show that the college-optimal stable matching rule implements the set of stable matchings via the Nash equilibrium concept. For any other stable matching rule the strategic behavior of the students may lead to outcomes that are not stable under the true preferences. By refining the Nash equilibrium (NE) concept, a general result shows that any stable matching rule implements the set of stable matchings via NE in the strong sense. Precise answers are given to the strategic questions raised.

Key words: stable matching, Nash equilibrium, mechanism, stable matching rule.

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INTRODUCTION

This paper deals with the cooperative model introduced in Sotomayor (1996-b), known nowadays as the school choice model. As in the College admission model of Gale and Shapley (1962), there is a set of students and a set of institutions, which can be taught of as being graduate centers (centers, for short) of Economics or Mathematics, to fix ideas. Each student can be enrolled in one center, at most. The institutions have quotas. The students have ordered lists of preferences over the institutions. It is assumed that a given procedure to evaluate the students is available for the centers. The students are then ranked by each center according to the result of this evaluation. This ranking defines the preference list of the center over the students. It is the case of a number of labor markets in the entrance level and admission markets of students to graduate centers in Economics and Business Administration in Brazil, in Turkey and in other countries.

For this model we consider the strategic game, called here Admission game, induced by a revelation mechanism defined by a stable matching rule. Due to the restriction on the preferences of the centers imposed by the market, these participants behave straightforwardly. Thus, a stable matching rule f selects a stable matching for the market $M(Q)=(S,C,Q,\{P_C,q_C\})$, where S and C denote the set of students and centers, respectively; Q is the profile of preference lists selected by the students; P_C and q_C are, respectively, the profile of the true preference lists and the array of quotas of the centers.

Sotomayor (1996-b) shows that every stable matching under the true preferences is a Nash equilibrium outcome for any stable matching rule[2]. The existence of Nash equilibria follows from the fact that stable matchings always exist (Gale and Shapley, 1962; Sotomayor, 1996-a, 1999). It is also proved in that paper that truth telling is not a Nash equilibrium unless the outcome produced for these preferences is the student-optimal stable matching. This means that, under any other stable matching rule, in equilibrium, at least one student will misrepresent his/her true preferences. The question is then *if, at equilibrium, when students behave strategically, the mechanism continues to yield stable matchings with respect to the true preferences, just as it does when the students reveal their true preferences.*

The present paper takes up this relevant question and manages to demonstrate some

[2]This result is a trivial adaptation of the existence theorem of Nash equilibria under the men-optimal stable matching rule for the Marriage model, due to Gale and Sotomayor (1985-b).

conclusive results. For the center-optimal stable matching rule we prove that *every Nash equilibrium outcome is stable under the true preferences.* This result plus the existence theorem of Nash equilibria of Sotomayor (1996-b) provides *an implementation of the set of stable matchings by the center-optimal stable matching rule via the Nash equilibrium concept.* For any other stable matching rule, however, the stability of the equilibrium outcomes may be lost. In fact, *for the student-optimal stable matching rule we show that the strategic behavior of the students may lead to outcomes that are not stable under the true preferences*[3].

Our main finding is that we can recover the stability of the equilibrium outcomes of any stable matching rule by refining the solution concept: a Nash equilibrium in the strong sense is a Nash equilibrium under every possible stable matching rule. This kind of refinement can be considered a natural solution concept when there is uncertainty about the matching that will be selected for a given strategy profile. The intuitive idea is that it consists of a profile of strategies that is, *ex post*, a Nash equilibrium under a random stable matching rule. For a given strategy profile this rule selects a stable matching at random (that is, with equal probability). This kind of rule is justified in centralized settings where neither of the center-optimal and student-optimal stable matching rules is considered socially fair. When the strategies are selected, no one knows which matching rule will be used. Then, *ex-ante,* the outcome of the game will be a lottery. Therefore, if the students play some profile of strategies Q whose set of stable matchings is given by, say, μ_1, μ_2 and μ_3, then they expect that each of these matchings will be selected with probability 1/3. It might be that Q is optimal for the students if μ_1 is selected but is not optimal for them if μ_2 or μ_3 is chosen. Thus, it is natural to expect that the students want to play a profile of strategies so that *ex-post,* after randomization is done, no one regrets the choice made. That is, if only one of the students had changed her strategy she would have had no chance of being better off. The solution concept that captures this idea of strategic equilibrium is that of Nash equilibrium in the strong sense[4]. This solution concept is justified by the

[3] For this same rule, but assuming that the students tell the truth and the colleges are allowed to behave strategically Roth (1985) has proved that there also may be Nash equilibrium outcomes that are unstable under the true preferences.

[4] This notion is the discrete version of the Nash equilibrium concept in the strong sense, introduced by Demange and Gale (1985) for the one-to-one matching model, in which the utilities are continuous in the money variable.

equivalence with the concept of Nash equilibrium for the random stable matching rule. This fact is proved here.

Taking this approach, we characterize the Nash equilibria in the strong sense as the strategy profiles which have a singleton set of stable matchings and are Nash equilibria of some stable matching rule. The idea is that, if x and y are stable matchings for the selected profile of strategies and some student prefers x to y then she can force the any stable matching rule to give her the mate she has at x, by pretending that only this center is acceptable to her, and so the selected strategy profile cannot be a Nash equilibrium for the matching rule that chooses y. It then happens that if the set of stable matchings for a Nash equilibrium preference profile is a singleton, then the Nash equilibrium outcome is the center-optimal stable matching for the selected profile. As we discussed above, such Nash equilibrium outcome must be stable under the true preferences. The desired implementation result is then obtained:

Any stable matching rule implements the set of stable matchings via the concept of Nash equilibrium in the strong sense and the student-optimal stable matching under the true preferences is the equilibrium outcome most preferred by the students.

The behavior of the coalitions is also considered here. If cooperation among students is possible, these agents might cooperate to influence the outcome towards their interest. In such cases the appropriate concept of equilibrium is that of strong equilibrium point due to Aumann (1959): *a profile of strategies such that no coalition of players can all gain by a simultaneous deviation (while the players outside the coalition maintain their strategies).* A special class of strategies used in practice by the participants consists of truncations of the true preferences[5]. For the case in which the students play, in equilibrium, truncations of the true preferences, we obtain additional results, which imply that: *the only Nash equilibrium outcome in the strong sense is the student optimal stable matching under the true preferences. Furthermore, a profile of truncations of the true preferences is a strong equilibrium point for every stable matching rule if and only if it determines a singleton set of stable matchings, whose only element is the student optimal stable matching under the true preferences.* As a consequence, *if the students only play*

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[5] The list of preferences Q(s) is a truncation of the true preferences P(s) at some agent v, if Q(s) ranks the centers in the same order as P(s), but ranks as unacceptable all centers which are ranked below v.

truncations of the true preference lists, then any stable matching rule implements the student optimal stable matching under the true preferences in strong equilibrium in the strong sense and in Nash equilibrium in the strong sense.

This paper is organized as follows. Section 2 describes the cooperative framework. In section 3 we describe the Admission games and state some known propositions needed for the proofs of our results. Section 4 presents an illustrative example. Section 5 is devoted to prove the results described above. Section 6 presents some final remarks. Appendix A demonstrates the equivalence between the Nash equilibrium in the strong sense concept and the Nash equilibrium concept for the random stable matching rule. For the sake of completeness Appendix B is devoted to prove some of the results of Sotomayor (1996-b) stated in section 3.

2. THE GRADUATE CENTER ADMISSION MODEL

The mathematical model for the Graduate center admission market is exactly the College admission model[6]. There are two finite and disjoint sets of participants, which we may think of as being a set of students, S, with m elements and a set of graduate centers, C (called centers, for short), with n elements. For each center c there is a positive integer number $q(c)$ called *quota* of c, which represents the number of positions it has to offer.

Each student has preferences over the centers and each center has preferences over individual students. We will assume these preferences are complete, transitive and strict, and so they may be represented by ordered lists, with $P(c)=s_1, s_3, s_2, c, s_4,\ldots$ denoting that center c prefers s_1 rather than s_3, that it prefers s_3 rather than s_2, that it prefers either one of them rather than leaving a position unfilled, and that all other students are unacceptable, in the sense that it would be preferable to leave a position unfilled rather than filling it with, say, student s_4. Similarly, $P(s)=c_1, c_2, s, c_3,\ldots$ represents the preferences of student s. We write $s >_{P(c)} s'$ to indicate that center c prefers s to s' and $s \geq_{P(c)} s'$ to mean c likes s at least as well as s', under the list of preference $P(c)$. Similarly, we write $c >_{P(s)} c'$ and $c \geq_{P(s)} c'$. Center c is *acceptable* to student s under P if $c >_{P(s)} s$, and student s is *acceptable* to center c if $s >_{P(c)} c$. We will usually abbreviate a preference list to include just the acceptable alternatives.

[6] See Roth and Sotomayor (1990) for an overview of this model.

We will assume that the preferences of a center over groups of students are *responsive* to its preferences over individual students. That is, if A is a set of students such that $|A| < q(c)$, and σ and σ' are in $S \cup \{c\}$ and are not in A, then center c prefers $A \cup \{\sigma\}$ to $A \cup \{\sigma'\}$ if and only if $\sigma >_{P(c)} \sigma'$. If $|A| \leq q(c)$ and $|B| \leq q(c)$, we will write $A >_{P(c)} B$ to denote that center c prefers A to B and $A \geq_{P(c)} B$ to denote that center c likes A at least as well as B.

The rules of the market allow that any student and center may mutually agree that the center will admit the student; any center may choose to keep any of its positions unfilled, and any student may remain unmatched if she wishes. A matching is an allocation of students to centers that respects the quotas of the agents. A student who is not matched to any center will be matched to himself/herself and will be called unmatched; a center that has some number of unfilled positions will be matched to itself in each of these positions. If μ is a matching, $\mu(s)=c$ denotes that student s is admitted by center c at the matching μ; $\mu(c)=\{s_2, s_4, c, c\}$, for example, denotes that center c, with quota $q(c)=4$, admits students s_2 and s_4 and has two unfilled positions; for $T \subseteq S \cup C$, $\mu(T)$ denotes the set of agents that are mates of some agent in T under μ. Given any two matchings μ and μ' and any agent $y \in S \cup C$ we write $\mu >_{P(y)} \mu'$ to mean $\mu(y) >_{P(y)} \mu'(y)$ and $\mu \geq_{P(y)} \mu'$ to mean $\mu(y) \geq_{P(y)} \mu'(y)$.

Definition 1. *A matching μ is* ***feasible*** *if for every pair (s,c), such that $\mu(s)=c$, s and c are mutually acceptable under P.*[7]

The adequate concept of stability for the matching models is that of setwise-stability (see Sotomayor (1999)). For the center admission model setwise-stability is equivalent to pairwise-stability, as proved in Roth (1985).

[7] For stability purposes, Definition 1 could require individual rationality instead of mutual acceptance. The reason for our choice is that under stable matchings, if two players form a partnership then they are mutually acceptable. Clearly, feasibility implies individual rationality, but the converse is not true.

Definition 2. *A matching* μ *is* ***pairwise-stable*** *(stable, for short) for the market* M *if it is feasible for* M *and there is no student* s *and center* c, *not matched to one another at* μ, *such that* a) $c >_{P(s)} \mu(s)$ *and* b) $s >_{P(c)} \sigma$, *for some* $\sigma \in \mu(c)$.

A student s and a center c, as in Definition 2, are said *to cause an instability in the matching* μ.[8]

The fact that setwise-stability is equivalent to pairwise-stability implies that stable matchings can be identified using only the preferences over individuals. That is, without knowing the responsive preferences that each center has over groups of individuals. Then, we will only consider the preferences of the centers over individual students. The profile of individual preferences of the participants will be denoted by $P=\{P_S,P_C\}$, where P_S is the profile of preference lists of the students and P_C is the profile of preference lists of the centers over individual students. Denoting by q_C the array of quotas of the centers, the market is then given by $M=(S,C,P_S,\{P_C,q_C\})$.

Definition 3. *For a given market* M, *a stable matching* μ_S *is called* ***the student- optimal stable matching*** *if* $\mu_S(s) \geq_{P(s)} \mu(s)$ *for every student* s *and for every stable matching* μ. *The* ***center-optimal stable matching*** μ_C, *is analogously defined.*

The existence of stable matchings as well as the existence of the student-optimal and the center-optimal stable matchings was first proved in Gale and Shapley (1962) by means of a well-known deferred-acceptance algorithm. A non-constructive proof for the existence of stable matchings is given in Sotomayor (1996-a). It can be easily seen that the set of stable matchings coincides with the strong core.

3. THE ADMISSION GAME

We consider the following matching mechanism for the Graduate center admission market. Given the original market $M(P)=(S,C,P_S,\{P_C,q_C\})$, each student $s \in S$ may replace his/her true preference list, $P(s)$, by any list of preference $Q(s)$. Let Q denote the profile

[8] For an abuse of language it is used, sometimes, the terminology " *(s,c)* blocks μ", which is already established in game theory to mean that this coalition can improve upon.

of lists of preference $Q(s)$, one list for each student $s \in S$. Once Q is selected, S, C, Q and $\{P_C, q_C\}$ are used as "input" for some algorithm that yields a stable matching for the market $(S,C,Q,\{P_C,q_C\})$, as a final "output". This procedure is described by a function f that we call *stable matching rule*. Since S, C and $\{P_C,q_C\}$ are fixed we denote such market by $M(Q)$ and denote by $E(Q)$ the set of stable matchings for $M(Q)$. For each profile of preferences Q, $f(Q)$ is the stable matching for $M(Q)$, which is selected by f.

Two special stable matching rules are the student-optimal stable matching rule and the center-optimal stable matching rule. Under the first one (respectively second one) the participants are assigned in accordance with the student-optimal (respectively center-optimal) stable matching for $M(Q)$.

The adoption of a stable matching rule f (or any mechanism with the stable matching rule f) for a given market $M(P)=(S,C,P_S,\{P_C,q_C\})$ induces a strategic game where the set of players is the set of students S; a strategy of student s is any list of preferences $Q(s)$; the outcome function is given by f and the true preferences of the players, called *sincere strategies*, are given by P_S. We say that Q *realizes* $f(Q)$ or that $f(Q)$ is determined by Q. We denote this game by $\Gamma(f)$ and we refer to it as the *admission game* induced by f. Sometimes we will refer to a stable matching for $M(Q)$ as a *stable matching under* Q.

The adequate concept of equilibrium is provided by that of Nash equilibrium.

Definition 4. *Let $\Gamma(f)$ be an admission game. The profile of strategies Q is a* ***Nash equilibrium*** *of $\Gamma(f)$ (or Q is a Nash equilibrium for the matching rule f) if for every student s, $f(Q) \geq_{P(s)} f(Q'(s),Q_{-s})$ for every strategy $Q'(s)$ of student s.*

That is, a set of strategies, one for each student, forms a Nash equilibrium if no student, by changing his/her strategy, can get a better payoff given the other students do not change their strategies. The existence of Nash equilibria follows from Proposition 4* proved in Appendix B.

If Q is a Nash equilibrium of $\Gamma(f)$ then we say that **$f(Q)$ is the Nash equilibrium outcome corresponding to Q.**

The following terminology will be used. We say that the list of preference $Q(s)$ is *a truncation of* $P(s)$ *at agent* v (v might be s), if $Q(s)$ ranks the centers in the same order as $P(s)$, but ranks as unacceptable all centers which are ranked below v. We say that $Q(s)$ *is a truncation of* $P(s)$ if it is a truncation of $P(s)$ at some agent v.

If Q is a profile of preferences for the students, $\mu_S(Q)$ is the student-optimal stable matching for $M(Q)$ and $\mu_C(Q)$ is the center-optimal stable matching for $M(Q)$. We set $\mu_S \equiv \mu_S(P_S)$ and $\mu_C \equiv \mu_C(P_S)$.

We will write Q_{-T} to indicate the restriction of Q to S-T, where $T \subseteq S$. If $T=\{s\}$ we will simply write Q_{-s} instead of $Q_{-\{s\}}$.

This paper also considers the strategic behavior of coalitions. The appropriate concept of equilibrium is that of strong equilibrium point due to Aumann (1959):

Definition 5. *Let* $\Gamma(f)$ *be an admission game. The profile of strategies* Q *is a* ***strong equilibrium point*** *of* $\Gamma(f)$ *if there is no coalition* S' *of students and a profile of strategies* Q' *such that* $f(Q'(s),Q_{-s}) >_{P(s)} f(Q)$ *for every* $s \in S'$.

The existence of strong equilibrium points follows from Proposition 5* proved in Appendix B. Our key concept is a refinement of the Nash equilibrium concept called here Nash equilibrium in the strong sense.

Definition 6. *A strategy profile* Q *is a Nash equilibrium in the strong sense for the admission game induced by some stable matching rule if for every student* s*, for every matching* μ *in* $E(Q)$*, for every* $Q'=(Q'(s),Q_{-s})$ *and every matching* μ' *in* $E(Q')$ *we have that* $\mu(s) \geq_{P(s)} \mu'(s)$.

That is, for every student s the worst matching in $E(Q)$ is weakly preferred to the best matching she could get by deviating from Q, assuming the other agents keep their strategies.

Clearly, if Q is a Nash equilibrium in the strong sense and f and f' are two different stable matching rules then Q is a Nash equilibrium under both of them. In other words, a *strategy profile* Q *is a Nash equilibrium in the strong sense if it is a Nash equilibrium for every possible admission game.*

In our context, this kind of refinement can be considered a natural solution concept when there is uncertainty about the matching that will be selected for a given strategy profile. This is what occurs under a *random stable matching rule.* As discussed in section 1, this is the name we give here to the procedure that picks, at random (i.e., with equal probability), a stable matching for the selected preferences. This game is well defined since the students know the probability distribution on the set of stable matchings for the strategy profile selected. The assumption that any stable matching has equal probability of being selected is natural in our context. Then, a Nash equilibrium in the strong sense is a Nash equilibrium *ex-post* of the game induced by the random stable matching rule.

Proposition 1 below, proved in Appendix A, shows that, indeed, this concept is equivalent to the Nash equilibrium concept for the game induced by the random stable matching rule. The concept of such equilibrium is the following. Let F be the random stable matching rule. Then for every profile of strategies Q, and for every μ in $E(Q)$, we have that $F(Q)=\mu$ with probability $1/|E(Q)|$. We denote this by writing $Prob(F(Q)=\mu)=1/|E(Q)|$.

Definition 7. *The profile of preferences* Q *is a Nash equilibrium of the random stable matching rule* F *if for every student* s *and strategy profile* $Q'=(Q'(s),Q_{-s})$ *we have that*

$$Prob(F(Q)=\mu) > Prob(F(Q')>_{P(s)} \mu) \text{ for every } \mu \text{ in } E(Q).$$

That is, for every μ in $E(Q)$, when s deviates and selects $Q'(s)$, the probability that the *random stable matching rule* F chooses some matching preferred to μ by s is less than the probability that F chooses μ when Q is selected. Thus, student s will only deviate if she has some chance to be better off. The condition above is equivalent to require that

A1. $|E(Q)|<|E(Q')|/|\{\mu'\in E(Q');\ \mu'>_s \mu\}|$, for every μ in $E(Q)$ and for all $Q'=(Q'(s),Q_{-s})$.

Remark 1. From A1 and the fact that $|E(Q)|\geq 1$ it follows that $|E(Q')| >|\{\mu'\in E(Q');\ \mu'>_{P(s)} \mu\}|$. Then, given s and any $Q'(s)$ there always exist some matching in $E(Q')$, say

μ^*, such that $\mu(s) \geq_{P(s)} \mu^*(s)$, so s does not profit from the deviation under some stable matching rule that yields μ when Q is selected and yields μ^* when Q' is selected. Then if Q is a Nash equilibrium of the game induced by F, Q is also a Nash equilibrium of some stable matching rule conveniently defined. ■

The equivalence between the two concepts can now be stated.

Proposition 1. *The profile of strategies Q is a Nash equilibrium of the game induced by F if and only if it is a Nash equilibrium in the strong sense.*

For the sake of completeness, we will present here the following propositions which will be needed for the proofs of our results. Proposition 3*-b, Proposition 4* and Proposition 5* will be proved in Appendix B.

Proposition 1*. (Gale and Sotomayor (1985-a)) *Consider the College admission market. Then, every student matched under some stable matching is matched under every stable matching; every college fills the same fraction of its quota at every stable matching.*

Proposition 2*. (Gale and Sotomayor (1985-a) Consider the *College admission market. Suppose $S' \subseteq S$ and let μ_S, μ_C, μ'_S,* and *μ'_C be the optimal stable matchings corresponding to (S,C,P,q_C) and (S',C,P',q_C), respectively, where P'_S agrees with P_S on S'. Then, for every center c,*

$$\mu_S(c) \succeq_{P(c)} \mu'_S(c) \text{ and } \mu_C(c) \succeq_{P(c)} \mu'_C(c).$$

Remark 2. *It follows from the proof of Proposition 2* that for all $s \in \mu_S(c)$ (respectively $s \in \mu_C(c)$), there is some $s' \in \mu'_S(c)$ (respectively $s' \in \mu'_C(c)$) such that $s \succeq_{P(c)} s'$.* ■

Proposition 3*. *Let f be a stable matching rule for the market $M(P)=(S,C,P_S,\{P_C,q_C\})$ with $f(P_S)=\mu$. Then:*

a) (Dubins and Freedman (1981) *If $\mu=\mu_S$ then f is collectively non-manipulable by the students.*

b) (Sotomayor (1996-b) *If* $\mu \neq \mu_S$ *then* f *is manipulable by at least one student. (This student is any* s *such that* $\mu_S(s) \neq \mu(s)$*).*

The existence of Nash equilibria in the strong sense is given by the following proposition:

Proposition 4*. (Sotomayor, 1996-b) *Let* μ *be a stable matching for the market* $M(P)=(S,C,P_S,\{P_C,q_C\})$*. Suppose each student* $s \in \mu(C)$ *chooses the strategy* $Q(s)$ *of listing only* $\mu(s)$ *as the only acceptable center;* $Q(s)=s$ *in case* s *is unmatched at* μ*. The profile of strategies* Q *is a Nash equilibrium in the strong sense and* μ *is the equilibrium outcome. Furthermore,* $E(Q)=\{\mu\}$*.*

The existence of strong equilibrium points in the strong sense is given by the following proposition:

Proposition 5*. (Sotomayor, 1996-b) *For each student* s*, let* $Q(s)$ *be a truncation of* $P(s)$ *at* $\mu_S(s)$*. Then* Q *is a strong Nash equilibrium under every stable matching rule. Furthermore* $\mu_{\underline{S}}$ *is the equilibrium outcome.*

Remark 3. ***If*** Q ***is a Nash equilibrium of some admission game in which*** Q ***realizes*** μ***, then*** μ ***is individually rational for the students under the true preferences, and so it is feasible.*** *In fact, if* $s >_{P(s)} \mu(s)$ *for some student* s*, then* s *can be better off by choosing* $Q'(s)=\{s\}$*, because then she will be unmatched at any stable matching for* $M(Q'(s),Q_{-s})$*. But this contradicts the assumption that* Q *is a Nash equilibrium of the given game.* The feasibility of μ then follows from the fact that P_C is the preference profile of the centers and μ is stable under $\{Q,P_C\}$. ■

4. EXAMPLE

Nash equilibria always exist, by Proposition 4*. One of the most desirable properties of an equilibrium outcome is its stability under the true preferences. The following example shows that the stability of the equilibrium outcome is not always reached even when the student optimal stable matching rule is adopted.

Example 1. (μ^1 *is a Nash equilibrium outcome but it is unstable under the true preferences*). Let $S=\{s_1,s_2,s_3\}$, $C=\{c_1,c_2,c_3\}$ and $q(c_j)=1$, for all $j=1,2,3$. Then, let P_S, P_C and Q be given by:

$P(s_1)= c_1, c_2, c_3$	$P(c_1)= s_2, s_1, s_3$	$Q(s_1)= c_2, c_3$
$P(s_2)= c_3, c_2, c_1$	$P(c_2)= s_3, s_1, s_2$	$Q(s_2)=c_3, c_1$
$P(s_3)= c_1, c_3$	$P(c_3)= s_1, s_3, s_2$	$Q(s_3)= c_1, c_3$

The stable matchings under Q are μ^1 and μ^2 where $\mu^1(s_1)=c_2$, $\mu^1(s_2)=c_3$, $\mu^1(s_3)=c_1$ and $\mu^2(s_1)=c_2$, $\mu^2(s_2)=c_1$, $\mu^2(s_3)=c_3$. Clearly , the matching μ^1 is the student optimal stable matching for $M(Q)$ and μ^2 is the center optimal stable matching for $M(Q)$.

Consider the matching rule f such that $f(Q)= \mu^1$. We claim that Q is a Nash equilibrium for $\Gamma(f)$. In fact, since μ^1 matches s_2 and s_3 to their first true choice, we only need to check the deviations of s_1. Then suppose by contradiction that there is some profile of preferences $Q'=(Q'(s_1),Q(s_2),Q(s_3))$ such that s_1 prefers $f(Q')$ to μ^1. Set $\mu'\equiv f(Q')$ Then $\mu'(s_1)=c_1$. Now, observe that, if $\mu'(s_2)=c_3$ then (s_3,c_3) causes an instability in μ' under Q'. If s_2 is not matched to c_3, s_2 is unmatched. But then (s_2,c_1) causes an instability in μ' under Q'. Thus, in any case we contradict the stability of μ' under Q'. Therefore, Q is a Nash equilibrium of every game in which Q realizes μ^1. It is a matter of verification that (s_1,c_1) causes an instability in μ^1 under the true preferences, so μ^1 is unstable under P_S. ■

In this example f is not necessarily the student-optimal stable matching rule. But Q is a Nash equilibrium of every game in which Q realizes μ^1, the student-optimal stable matching for $M(Q)$. Thus, if the student-optimal stable matching rule is used and the students do not play their sincere strategies, the equilibrium outcome may be unstable under the true preferences.

5. THE IMPLEMENTABILITY OF THE SET OF STABLE MATCHINGS

In this section we prove the results described in section 1, which aim to recover the stability of the equilibrium outcomes and to provide some implementation of the core. Theorem 1 shows that the equilibrium outcome yielded by the center-optimal stable matching rule is always stable under the true preferences. By combining this result with Proposition 4* we get a strong core implementation result. For its proof we need lemmas 1 and 2, which are important *per se.* The first one implies, in particular, that *if the profile of true preferences, P, is a Nash equilibrium under a given stable matching rule then the equilibrium outcome must be the student optimal stable matching.* Then, if the outcome produced by the true preference profile under some stable matching rule is not the student optimal stable matching, the mechanism is manipulable by at least one student. Thus we obtained an alternative proof of Proposition 3*-b first proved in Sotomayor (1996-b). (See Appendix B).

Lemma 1. *Let Q be a Nash equilibrium of the game $\Gamma(f)$. Let $f(Q)=\mu$. Then $\mu(s)\geq_{P(s)}\mu'(s)$ for every μ' in $E(Q)$.*

Proof. Suppose by way of contradiction that $\mu'(s) >_{P(s)} \mu(s)$ for some $s\in S$ and $\mu'\in E(Q)$. The individual rationality of μ implies that $\mu'(s)\equiv c$ for some $c\in C$. Let $Q'(s) = c, s$ and $Q'_{-s} = Q_{-s}$. It is clear that μ' is stable under Q'. By Proposition 1*, s is matched at every stable matching under Q', so s gets c under f when the students play Q', so s gets better by deviating from Q, and so Q cannot be a Nash equilibrium of the game $\Gamma(f)$, which is a contradiction. ■

Lemma 1 implies that if Q is a Nash equilibrium and it is a truncation of the true preferences then $\mu(s)\geq_{Q(s)}\mu'(s)$ for every μ' in $E(Q)$, so $\mu=\mu_S(Q)$. Consequently, under the center-optimal stable matching rule we must have that $\mu=\mu_S(Q)=\mu_C(Q)$ and so $E(Q)$ is a singleton. Theorem 1 will then imply that μ is stable under the true preferences.

The second lemma is mathematically unusual. It provides a way of concluding that a strategy profile Q is not a Nash equilibrium of all admission games for $M(P)$, based only on the direct examination of the size of the core for $M(Q)$, without necessity of knowing the true preferences of the students. Since the definition of a Nash equilibrium involves the true preferences of the players, this is surprising. In fact, we do not know of any comparable result concerning the strategic equilibrium of a game.

This lemma *characterizes the Nash equilibrium strategies in the strong sense* as being those preference profiles which have a singleton set of stable matchings and are Nash equilibria of some stable matching rule. Then, if μ is the only stable matching for $M(Q)$, and the students select Q, they will be sure that they will get μ under any stable matching rule. If Q is a Nash equilibrium under some stable matching rule, it will be a Nash equilibrium in the strong sense. Then, we obtain the existence of strategy profiles that are Nash equilibria for every admission game, independent of the stable matching rule.

Lemma 2. *Let Q be a Nash equilibrium of some admission game. Then Q is a Nash equilibrium of every admission game if and only if $|E(Q)|=1$.*

Proof. Suppose Q is a Nash equilibrium of every admission game. If μ and μ' are stable matchings for Q, then Q is a Nash equilibrium of the games $\Gamma(f)$ and $\Gamma(f')$, where $f(Q)=\mu$ and $f'(Q)=\mu'$. In this case, Lemma 1 plus the strictness of the preferences would imply that $\mu=\mu'$. Hence, $|E(Q)|=1$.

In the other direction, suppose that Q is a Nash equilibrium of $\Gamma(f)$, with $f(Q)\equiv\mu$ and $E(Q)=\{\mu\}$. Now, suppose Q is not a Nash equilibrium of the admission game $\Gamma(f')$, for some stable matching rule f'. Since $E(Q)=\{\mu\}$, we have that $f'(Q)=\mu$. Then there is some student s and $Q'=(Q'(s),Q_{-s})$, such that $f'(Q') >_{P(s)} \mu$. Set $\mu'\equiv f'(Q')$. By Remark 2, μ is individually rational for M, so $\mu'(s) >_{P(s)} \mu(s) \geq_{P(s)} s$, which implies that s must be matched to some c at μ'. Hence

$$c >_{P(s)} \mu(s) \qquad (1)$$

Let $Q''(s)=c$ and $Q''_{-s}=Q_{-s}$. It is clear that μ' is stable under Q''. Then, by Proposition 1*, s is matched at every stable matching under Q'', so s gets c under f when the students play Q''. But then (1) implies that the deviation $Q''(s)$ is profitable to s in the game $\Gamma(f)$, which contradicts the assumption that Q is a Nash equilibrium for $\Gamma(f)$. Hence Q is a Nash equilibrium in every game as we wanted to show. ■

Of course, not all Nash equilibria have a singleton set of stable matchings. In Example 1, Q is a Nash equilibrium when it realizes μ^1 and $M(Q)$ has two stable matchings. In this example, Q is not a Nash equilibrium when it realizes μ^2 (s_2 can profit by deviating and selecting $Q'(s_2)=c_3$).

Theorem 1. *Let* Q *be a preference profile. Let* f *be a stable matching rule such that* $f(Q)=\mu_C(Q)$. *If* Q *is a Nash equilibrium for* $\Gamma(f)$ *then* $\mu_C(Q)$ *is stable for the market M(P). Furthermore, the center-optimal stable matching rule implements the set of stable matchings via the Nash equilibrium concept.*

Proof. Let f be a stable matching rule such that Q is a Nash equilibrium of $\Gamma(f)$ and $f(Q)=\mu_C(Q)$. For simplicity of notation set $\mu_C(Q)\equiv\mu$. Suppose, by way of contradiction, that μ is not stable under P_S. Since μ is feasible under the true preferences by Remark 3, then there exists a pair (s,c) that causes an instability in μ and so

$c>_{P(s)}\mu(s)$ and $s>_{P(c)}\sigma$, for some $\sigma\in\mu(c)$ (σ may be c). (1)

We will show that Q is not a Nash equilibrium of $\Gamma(f)$, which is a contradiction. To see this, let s deviate from $Q(s)$ by listing only c on her list of acceptable centers. Set $\mu'\equiv f(Q')$, where Q' differs from Q only in the new list of s. If s does not get c, then she is unmatched at μ'. By Proposition 1*, s is also unmatched at $\mu'_C\equiv\mu_C(Q')$.

On the other hand, the stability of μ'_C in $M(Q')$ and the fact that $\mu'_C(s)=s$ imply that center c fills its quota under μ'_C and

$s'>_{P(c)}s,\ \forall s'\in\mu'_C(c)$. (2)

By (1), (2) and the transitivity of the preferences,

$$s'>_{P(c)}\sigma,\ \forall s'\in\mu'_C(c). \quad (3)$$

Now, observe that the restriction of matching μ'_C to the market $M^*=(S^*,C,Q_{-s},\{P_C,q_C\})$, with $S^*\equiv S-\{s\}$, is stable for this market. Let μ_C^* denote the center-optimal stable matching for M^*. Then, $\mu_C^*(c)\geq_{P(c)}\mu'_C(c)$, by the optimality of μ_C^*. Since c fills its quota at μ'_C, Proposition 1* implies that

$$c \text{ fills its quota at } \mu_C^*. \quad (4)$$

On the other hand, from Proposition 2* applied to M^* and $M(Q')$ it follows that $\mu_C^*(c)\leq_{P(c)}\mu'_C(c)$, so $\mu_C^*(c)=\mu'_C(c)$. Therefore, using (3),

$$c \text{ prefers any student in } \mu_C^*(c) \text{ rather than } \sigma. \quad (5)$$

From Proposition 2* applied to M^* and $M(Q)$ it follows that $\mu(c) \geq_{P(c)} \mu_C^*(c)$, and from Remark 2 it follows that center c weakly prefers σ to some of its mates in $\mu_C^*(c)$. By (4), all c's mates at μ_C^* are students, so c weakly prefers σ to some student in $\mu_C^*(c)$, which contradicts (5). Hence $\mu'(s)=c$, so Q is not a Nash equilibrium of the game $\Gamma(f)$, as we wanted to show. ■

Theorem 2 provides us with a strong core equivalence via Nash equilibrium in the strong sense. For its proof we need the following lemma.

Lemma 3. Let *f be any stable matching rule. Let Q be a Nash equilibrium of the game $\Gamma(f)$. If $|E(Q)|=1$ then the corresponding Nash equilibrium outcome is stable under* P_S.
Proof. If $E(Q)$ has only one element than this element is the optimal stable matching for the centers for market $M(Q)$. By Theorem 1 this matching must be stable under the true preferences. ■

This result asserts that a sufficient condition for the stability under the true preferences of the Nash equilibrium outcomes is that the set of stable matchings for the corresponding Nash equilibria is a singleton. Since the set of stable matchings for a Nash equilibrium in the strong sense has a singleton core, it follows that the equilibrium outcomes yielded by such equilibria are always stable under the true preferences. Then we can get the following:

Theorem 2. *Let Q be a Nash equilibrium in the strong sense. Then the corresponding Nash equilibrium outcome is stable under the true preferences. Moreover, any stable matching rule implements the set of stable matchings in Nash equilibria in the strong sense and μ_S is the equilibrium outcome most preferred by the students .*
Proof. Immediate from Lemmas 2 and 3. For the other assertion use Proposition 4*. ■

Theorem 2 implies that if Q is a Nash equilibrium in the strong sense then $\mu_S(Q)$ is stable under the true preferences. Proposition 2 adds that indeed $\mu_S(Q)=\mu_S$ if Q is obtained via truncation of the true preferences:

Proposition 2. *For each student* s *let* $Q(s)$ *be a truncation of* $P(s)$. *If the profile of strategies* Q *is a Nash equilibrium in the strong sense then* $E(Q)=\{\mu_S\}$.

Proof. If Q is a Nash equilibrium in the strong sense then $|E(Q)|=1$ by Lemma 2. Then set $E(Q)\equiv\{\mu_S(Q)\}$. Theorem 2 implies that $\mu_S(Q)$ is stable for M, so the optimality of μ_S in $M(P)$ implies $\mu_S(s) \geq_{P(s)} \mu_S(Q)(s)$ $\forall s \in S$, so $\mu_S(s) \geq_{Q(s)} \mu_S(Q)(s)$ $\forall s \in S$, by construction of Q, and so μ_S is stable for $M(Q)$. But then, $\mu_S(Q)=\mu_S$ and we have completed the proof. ■

Proposition 3 characterizes the truncations of P_S that are strong equilibrium points in the strong sense (strong equilibrium points of every stable matching rule). Formally,

Proposition 3. *For each student* s *let* $Q(s)$ *be a truncation of* $P(s)$. *Then* Q *is a strong equilibrium point in the strong sense if and only if* $E(Q)=\{\mu_S\}$.

Proof. The proof that the condition is necessary follows from Proposition 2. To see that the condition is sufficient, suppose $E(Q)=\{\mu_S\}$. Then $\mu_S=\mu_S(Q)$. Let f be any stable matching rule. Suppose by way of contradiction that there are $S'\subseteq S$ and $Q'=(Q'_{S'},Q_{-S'})$ such that $f(Q'(s)) >_{P(s)} f(Q(s))$ for every $s \in S'$. Then, $f(Q'(s)) >_{Q(s)} f(Q(s))$ by construction of $Q(s)$. But $f(Q)= \mu_S(Q)$, which contradicts Proposition 3*-a applied to $M(Q)$. Hence Q is a strong Nash equilibrium for f. Since f is arbitrary, it follows that Q is a strong Nash equilibrium in the strong sense and the proof is complete. ■

Observe that the strategy Q in Example 1 is also a strong equilibrium point under the student-optimal stable matching rule. The strong equilibrium outcome is not μ_S , which contradicts Proposition 10 of Romero-Medina (1998) that asserts that the student-optimal stable matching rule implements the student-optimal stable matching under the true preferences in strong equilibrium. However we have that:

Corollary 1. *Suppose the students only play truncations of the true preference lists. Let* f *be any stable matching rule. Then* f *implements the student-optimal stable matching under* *the true preferences in strong equilibrium in the strong sense and in Nash equilibrium in the strong sense.*

Proof. For every student s let $Q(s)$ be the truncation of $P(s)$ at $\mu_S(s)$. We claim that $E(Q)=\{\mu_S\}$. In fact, μ_S is stable for $M(Q)$ and any other stable matching μ for $M(Q)$ will be unstable for $M(P)$ by the optimality of μ_S. However, any pair that blocks μ under P_S will also block μ under Q, by the construction of Q, which is a contradiction. Then $E(Q)=\{\mu_S\}$ and so $f(Q)=\mu_S$. By Proposition 3, Q is a strong equilibrium point in the strong sense, so it is also a Nash equilibrium in the strong sense. On the other hand, every strong equilibrium in the strong sense given by truncations of P_S realizes μ_S, also by Proposition 3, and every Nash equilibrium in the strong sense given by truncations of P_S realizes μ_S by Proposition 2. Hence we have completed the proof.■

6. FINAL REMARKS

The central issue of this paper is the stability of the equilibrium outcome under a stable matching rule. To derive our results we restrict the College admission model to a Graduate center admission model and refine the solution concept. Under this restriction centers behave straightforwardly and the strategies of the students are given by their preferences over the centers. The outcome function is defined by a stable matching rule, not necessarily the student-optimal or the center-optimal stable matching rule.

Suppose that the profile of true preferences has more than one stable matching. If the stable matching rule selects the optimal stable matching for the students when they behave straightforwardly, Proposition 3*-a shows that the respective mechanism is non-manipulable by these agents. This means that under the student-optimal stable matching rule the sincere strategy profile is a Nash equilibrium, so the stability of the equilibrium outcome under the true preferences is reached in this case. In any other case, Proposition 3*-b proves that the mechanism is manipulable by at least one student. This suggests that, in equilibrium, the candidates misrepresent their preferences, thereby the stability of the outcome produced may not be guaranteed. In fact, if the students behave strategically, even the student-optimal stable matching rule may produce equilibrium outcomes that are unstable under the true preferences. This was the case illustrated in the example presented here. Student s_1 is assigned to center c_2 even though she and center c_1 would prefer each other to their equilibrium partners. A deviation by student s_1 does not pay in this game, because the other students' preference lists generate the implicit threat to switch to a stable matching in which student s_1 either continues with c_2 or gets c_3, his worst alternative.

However, the instability of the equilibrium outcomes according to the true preferences does not occur in every stable matching rule. We show that the equilibrium outcome is always stable under the true preferences if the center-optimal stable matching rule is used. Proposition 4* is then employed to prove the desired equivalence of the strong core and the equilibrium outcomes.

A very simple sufficient condition for the stability of the equilibrium outcomes is then found: if the set of stable matchings for a Nash equilibrium preference profile is a singleton, such Nash equilibrium outcome is stable under the true preferences.

When neither of the center-optimal and student-optimal stable matching rules is considered socially fair, it is reasonable to assume that a random stable matching rule F selects a matching at random, that is, with equal probability, among the stable matchings for the revealed preference profile. A refinement of the concept of Nash equilibrium was introduced as being an ex-post (after randomization is done) Nash equilibrium of F. Such equilibrium concept, called Nash equilibrium in the strong sense, is a Nash equilibrium of every possible admission game for a given center admission market. This concept was then shown to be equivalent to the concept of Nash equilibrium of the random stable matching rule. We characterized such equilibria as the Nash equilibria for any particular admission game that has a singleton set of stable matchings. Therefore, our sufficient condition for the stability of the equilibrium outcomes is satisfied when the students play Nash equilibrium in the strong sense. Finally, we demonstrated that every stable matching rule implements the set of stable matchings via the Nash equilibrium in the strong sense concept. Equivalently, the random stable matching rule implements the set of stable matchings via Nash equilibria.

Under the hypothesis that the students only play truncations of the true preferences we proved that any stable matching rule implements the student-optimal stable matching via the Nash equilibrium in the strong sense concept and via the strong equilibrium in the strong sense concept.

The implementability of the core correspondence through stable mechanisms, which are closely related to Gale and Shapley's algorithms, was also investigated by several authors. We can cite, among others, Gale and Sotomayor (1985-b), Roth (1984), Alcalde (1996), Alcalde and Romero-Medina (2000), Alcalde, Perez-Castrillo and Romero-Medina (1998) and Sotomayor (2002). The common feature of these mechanisms is that the equilibrium outcomes are stable under the true preferences. Sotomayor (2003) proved that this property also holds for a mechanism, which is not designed for producing stable outcomes. Romero-Medina (1998) considers that colleges have fixed preferences over students and limits the number of options that students can declare. He focuses only on the student-optimal stable matching rule. However, Example 1 of the present work shows that some main results of that paper, namely Theorem 7, Corollary 8 and Proposition 10 are not correct.

All the results presented here were proved by the first time by the author in previous versions that have been widely circulated since 1993. The strategic model treated here was first considered in Sotomayor (1996-b)[9], motivated by the admission market for Graduate Schools of Economics in Brazil and had the purpose to investigate the strategic behavior of the students. In this market, the students are submitted to five tests: Mathematics, Statistics, Microeconomics, Macroeconomics and a test about the Brazilian Economy. The schools attribute weights to each one of the five subjects. The quotas and weights of the centers and the students' scores in the tests are common knowledge. The students are evaluated by the centers in accordance to their weighted average score in the tests. These evaluations are used by the centers to determine a ranking of the students.

This first version is entitled "Admission mechanisms of students to colleges. A game-theoretic modeling and analysis" and was written in Portuguese. This paper was presented in 1997 in several seminars (University Autonoma and Pompeu Fabra, in Barcelona, State University of New York at Stony Brook, University of Pittsburgh, Institute of Applied and Pure Mathematics of Rio de Janeiro, among others). Due to the frequent requests for a copy of the paper a second version, in English was written in 1998. In its

[9] For the center optimal stable matching rule, Sotomayor (1996) also characterizes all the dominated strategies for a given student.

preparation we found out example 1.[10] This was very surprising since the stability of the equilibrium outcome had always been obtained in the previous approaches for the one-to-one matching models, discrete and continuous. This example was the motivation for the use of Nash equilibrium in the strong sense to recover stability and to get our two implementation results. This version also contains the results on the strong equilibrium points.

This second version, entitled "The strategy structure of the college admissions stable mechanisms," was presented in several congresses: First World Congress of the Game Theory Society (Bilbao, 2000), International Symposium of Mathematics Programming (Atlanta, 2000), World Congress of the Econometric Society (Seattle, 2000) and Jornadas Latino Americanas de Teoria Economica (San Luis, Argentina, 2001). The abstract was published in the Annals of these congresses.

A third version, entitled "Reaching the core through college admissions stable mechanisms," with a different *format* was prepared and presented in the International Conference on Game Theory (Stony Brook, 2001), Brazilian Meeting of Econometrics, (Salvador, Brazil, 2001) and Latin American Meeting of the Econometric Society (Buenos Aires, Argentina, 2001). The abstract was also published in the Annals of these congresses.

Other authors have worked in the College admission model under several different approaches. The model which motivated the original version of this paper can then be viewed as the so called school choice model. For an overview on the most recent achievements on the School choice problem see Abdulkadiroglu and Sonmez (2003), Balinski and Sonmez (1999) and Kesten (2006-a,b), among others.

[10] This example indicated to us that some incorrect conclusion on the stability of the Nash equilibrium outcomes had been reached in Theorem 4 of Sotomayor (1996-b). Readers of that paper are advised that they need to substitute "stable mechanism" for "college-optimal stable mechanism".

APPENDIX A.

In this section we will present the proof of Proposition 1. The following lemma will be used in the proof. It implies that if the set of stable matchings for some profile of strategies Q has more than one element then Q is not a Nash equilibrium of F. The idea is that if μ and μ' are in $E(Q)$ and some student s prefers μ' to μ then she can force the random stable matching rule F to give her $\mu'(s)$ by deviating and stating $Q'(s)=\mu'(s)$.

Lemma A. *If Q is a Nash equilibrium of the game induced by the random stable matching rule F then $|E(Q)|=1$.*

Proof. In fact, for suppose it is not. Then we can choose μ and μ' in $E(Q)$ and s such that $\mu' >_{P(s)} \mu$. Clearly, any stable matching for the profile of strategies $Q'=(Q'(s),Q_{-s})$, where $Q'(s)=\mu'(s)$, matches s with her mate under μ'. Therefore $|E(Q')|/|\{\mu'' \in E(Q'); \mu'' >_{P(s)} \mu\}|=1$, and so, by using expression (A1) we get that $|E(Q)| <1$, absurd. Hence $|(E(Q)|=1$ and the proof is complete.■

Proposition 1. *The profile of strategies Q is a Nash equilibrium of the game induced by F if and only if it is a Nash equilibrium in the strong sense.*

Proof. If Q is a Nash equilibrium of the game induced by F we have that $|(E(Q)|=1$ by Lemma A. Let $E(Q)=\{\mu\}$. By Remark 1, Q is the Nash equilibrium of some stable matching rule conveniently defined. Then, Lemma 2 implies that Q is a Nash equilibrium in the strong sense. Conversely, if Q is a Nash equilibrium in the strong sense then, for every student s, for every μ in $E(Q$, for every $Q'=(Q'(s),Q_{-s})$ and every μ' in $E(Q')$ we have that $\mu(s) \geq_s \mu'(s)$. Then if s deviates the probability that the *random stable matching rule* selects a matching strictly preferred to μ by s is zero. Hence, Q is a Nash equilibrium of F.■

APPENDIX B

This section is devoted to prove the results of Sotomayor (1996-b) used in this paper. Proposition 3*-b is an immediate corollary of Lemma 1. We will present its original proof for the sake of the insights it provides.

Proposition 3*-b. *Let f be a stable matching rule for the market $M(P)=(S,C,P_S,\{P_C,q_C\})$ with $f(P_S)=\mu$. Then, if $\mu\neq\mu_S$ then f is manipulable by at least one student. (This student is any s such that $\mu_S(s)\neq\mu(s)$).*

Proof. Since $\mu\neq\mu_S$, there is at least one student s such that

$$\mu_S(s) >_{P(s)} \mu(s) >_{P(s)} s, \qquad (1)$$

where the last inequality is due to Proposition 1* (if $\mu(s)$=s, then $\mu_S(s)$=s=$\mu(s)$, contradiction). Let $Q(s)$ be the truncation of $P(s)$ at $\mu_S(s)$. Set $\mu' \equiv f(\{Q(s),P_{-s}\})$. We claim that $\mu'(s) \geq_{P(s)} \mu_S(s)$. In fact, first observe that μ_S is stable for $M'=(S,C,\{Q(s),P_{-s}\},\{P_C,q\})$. Thus, since s is matched at μ_S and μ' is stable for M', Proposition 1* implies that s will be matched at μ'. Hence, $\mu'(s) \geq_{P(s)} \mu_S(s)$, by the construction of $Q(s)$. Using (1) we get that $\mu'(s) >_{P(s)} \mu(s)$. Hence s can improve her payoff by misrepresenting her preferences.■

Hence, if f is not collectively manipulable, it is not individually manipulable, so $f(P_S) =\mu_S$.

Proposition 4*. *Let μ be a stable matching for the market $M(P)=(S,C,P_S,\{P_C,q_C\})$. Suppose each student $s\in\mu(C)$ chooses the strategy $Q(s)$ of listing only $\mu(s)$ as the only acceptable center; $Q(s)=s$ in case s is unmatched at μ. The profile of strategies Q is a Nash equilibrium in the strong sense and μ is the equilibrium outcome. Furthermore, $E(Q)=\{\mu\}$.*

Proof. It is clear that μ is stable for the market $M(Q)=(S,C,Q,\{P_C,q\})$. Furthermore, μ is the only stable matching for $M(Q)$, for any other matching would leave some student s in

$\mu(C)$ unmatched, which is not possible by Proposition 1*. Hence

μ **is the matching produced by any stable matching rule when the students play** $\boldsymbol{Q}$**.** (1)

To see that Q is a Nash equilibrium in the strong sense, suppose by the way of contradiction that some student s changes her strategy from $Q(s)$ to $Q'(s)$, yielding a new set of strategies $Q'=\{Q'(s), Q_{-s}\}$. Suppose further that there are f, $\mu' \in E(Q')$ and c, with $f(Q')=\mu'$ and $\mu'(s)=c$, such that $c >_{P(s)} \mu(s)$. Then, center c must have filled its quota under μ and prefers any student assigned to it at μ rather than student s, for if not (s,c) would block μ in $M(P)$. But then, some student s', who would have been assigned to c at μ, would not have been assigned to c at μ'. But if this were so, since c was the only acceptable mate to s', it would follow that s' would be unmatched at μ', and so (s',c) would have blocked μ' in $M(Q')$, which is a contradiction. Hence Q is a Nash equilibrium under any stable matching rule, so it is a Nash equilibrium in the strong sense. Now use (1) to get that μ is the Nash equilibrium outcome corresponding to Q.■

The proof of Proposition 5* presented below is simpler than the one presented in Sotomayor (1996-b).

Proposition 5*. *For each student* s, *let* $Q^*(s)$ *be a truncation of* $P(s)$ *at* $\mu_S(s)$. *Then* Q^* *is a strong Nash equilibrium under every stable matching rule. Furthermore* μ_S *is the equilibrium outcome.*

Proof. We claim that μ_S is the only stable matching for $M(Q^*)=(S,C,Q^*,\{P_C,q\})$, for clearly μ_S is stable for $M(Q^*)$ and any other stable matching μ' must have $\mu'(s)\neq\mu_S(s)$ for some s, and so Proposition 1* implies that s is not unmatched at μ'. Then $\mu'(s) >_{P(s)} \mu_S(s)$ by construction of Q^*. Since μ_S is the student-optimal stable matching for P, this means μ' is unstable under the true preferences, so it is blocked by some pair (s', c). By construction of Q^*, (s',c) blocks μ' under Q^* preferences, contradicting the stability of μ' under Q^*.

Therefore, $E(Q^*)=\{\mu_S\}$ and then Proposition 3 implies that Q^* is a strong Nash equilibrium in the strong sense. Clearly, μ_S is the equilibrium outcome. ■

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