SOME NEW CHARACTERIZATION OF THE CORE ALLOWING MEANINGFUL COMPARATIVE STATICS IN A HYBRID JOB MATCHING WITH HETEROGENEOUS FIRMS AND WORKERS

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ABSTRACT

Flexible firms compete by means of wages in the Assignment market while rigid firms have no flexibility over terms of appointment in the Marriage market. Workers trade with both kinds of firms in the hybrid market.

Examples show that standard results that characterize the core of the Marriage market (respectively, Assignment market) are not robust to the entrance of flexible (respectively, rigid) firms to this market. A new algebraic structure provides a different characterization for the core of the hybrid model and reflects a sort of robustness to the exit of rigid (respectively, flexible) firms from this market. Meaningful comparative static results are derived.

Key words: stable matching, core, firm-optimal stable outcome, worker-optimal stable outcome, comparative static.
JEL classification: C78, D78

INTRODUCTION

As observed by Cournot, Edgeworth, Bohm-Bawerk and other authors, two-sided market models are important, not only for the insights they can provide into economic situations with many types of agents, but also because, in real situations, most transactions are actually bilateral. A special class of these markets is that of matching markets. The theory for such markets started with Gale and Shapley (1962), who introduced the Stable Matching problem for the Marriage and the College Admissions markets, as a rare instance of an exercise in “pure” mathematics (combinatorial theory of ordered sets). In 1972, Shapley and Shubik introduced money into the Marriage model, establishing the Assignment game. Over the years the stable matching problem has been generalized to several two-sided matching models, which have been widely modeled and analyzed under the cooperative and non-cooperative game theoretic approaches. Through these models a variety of markets has become better understood, which has considerably contributed to their organization.

The two-sided matching markets have as primary object the formation of partnerships between agents on one side with agents on the opposite side. For a wide class of such markets, if a partnership is formed, the partners undertake some activity together, which produces a gain that is split between them. A special subclass is that of job markets with heterogeneous and well-informed firms and workers: If a firm hires a worker, the productivity of the pair is allocated into salary for the worker and profit for the firm. In some markets the salary can be negotiated between the two agents; in some other markets salaries are part of the job description, and are only one of the factors that determine the preferences that workers have over firms. If we observe the market for academic positions and professors, for example, we would see that while the American Universities compete with each other in terms of salaries, the French public Universities offer a preset and fixed salary. Traditionally, the former kind of market has been modeled as a continuous matching market, whose simplest form is the Assignment Game. Salary may be negotiated and may vary continuously on the set of real numbers. The latter has been modeled as the discrete matching market known as Marriage Market. The key notion in both markets is that of stability. In the Marriage Market, a matching is stable if it is individually rational and if no firm and no worker prefer each other to their current partners. For the Assignment Game, an outcome is a matching plus a payoff vector, with one payoff for each agent. It is stable if it is individually rational and if no firm and worker can negotiate a payoff which is higher than the payoff they obtain from the current outcome. In both models the set of stable outcomes coincides with the core.

The main features that characterize the core outcomes of these models are the following:

(A) *The set of stable payoffs is a non-empty complete lattice*[2] *under the partial order induced by the preferences of the workers and under that induced by the preferences of the firms*[3].

(B) *There is an opposition of interests between the two sides of the market along the whole set of stable outcomes: If* x *and* y *are two core outcomes and all workers prefer* x *to* y *then all firms prefer* y *to* x.

(C) *If some agent is unmatched under some stable outcome of the Marriage market, then this agent continues to be unmatched under any other stable outcome of this market (Gale and Sotomayor, 1985-a). If some agent is unmatched under some stable outcome of the Assignment market, then this agent receives zero payoff under any other stable outcome of this market (Demange and Gale, 1985).*

The complete lattice structure is of economic interest, because it guarantees the existence of two extreme points that have an important meaning for the model. These points reflect a surprising coincidence of interest among agents on the same side of the market, and a corresponding conflict of interest among agents on opposite sides. All workers, as well as all firms, agree on the best stable payoff for them. In addition, the best core outcome for one side is the worst core outcome for the other side. These outcomes are called worker-optimal and firm-optimal stable payoffs, respectively. Their existence is of great applicability in the organization of real matching markets, because

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[2] A lattice is a partially ordered set any two of whose elements have a supremum and an infimum. A lattice is complete when each of its subsets has a supremum and an infimum. (See Birkhoff, 1973).

[3] The lattice property of the core for the Assignment market was originally obtained by Shapley and Shubik (1972). They also showed that there is a polarization of interests between the two sides of the market along the whole core. With regard to the Marriage market with strict preferences, the original proof of that mathematical structure for the set of stable matchings is due to Conway and the existence of that polarization of interests to Knuth (1976). A different proof of both results was given by Gale and Sotomayor (1985-a). This proof is also presented in Roth and Sotomayor (1990) where the reader can find an overview of these two models.

of the close relation between the mechanism that yields the optimal stable outcome for one side and its non-manipulabilitiy by the agents of that side. In a different context, meaningful comparative static results can be derived by allocating the agents according to a firm-optimal or a worker-optimal stable outcome (see Gale and Sotomayor, 1985-b and Demange and Gale, 1985).

Property (B) is inherent to the core of the majority of matching models. Property (C) is certainly of interest to a market organizer who faces the problem of choosing between a firm-optimal stable mechanism and a worker-optimal stable mechanism. Although there is a discrepancy between the two procedures, this fact is not relevant for the allocation of "bad" workers or "bad" firms, since these agents would have the same payoffs under both outcomes. In parallel, the "good" agents would be allocated at both outcomes with a positive payoff.

In practice, however, the hypothesis that agents’ choices are restricted to only one market (discrete or continuous) is, sometimes, a hindrance to seeing the discrete and continuous models as real job market models. In fact, a wide range of real-world matching markets are hybrid. In the United States, for example, new law school graduates may enter the market for associate positions in private law firms, which negotiate salaries, or they may seek employment as law clerks to federal circuit court judges, which are civil service positions with predetermined fixed salaries. In Brazil, new professors may enter the market for tenure positions (with preset and fixed salaries) in federal Universities, or they may seek employment in private Universities, which do not offer tenure, but compensate with better and negotiable salaries. All Brazilian Graduate Centers of Economics offer a fellowship to an entrant graduate student with the same stipend to all students. In the admission process, some private Universities negotiate an additional amount of money with the students they are willing to admit.

In Eriksson and Karlander (2000), a symmetric one-to-one matching model[4] is proposed to unify the discrete and continuous models: Some agents are “rigid” as to the payoff sharing between partners, while others are “flexible”. The Marriage Market is obtained when all agents on one side are rigid and the Assignment Game results when no participant is rigid.

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[4] Roth and Sotomayor (1996) proposed a unified model for the Marriage model and the continuous model of Demange and Gale (1985) that offers common proofs for basic results.

For this model, Sotomayor (2000) proved that the set of stable payoffs has the same characterization as the core of the *non-hybrid models*[5], given by (A), (B) and (C) above, under the assumption that the core defined by strong domination coincides with the core defined by weak domination.

Without seeking wide generality, the present paper deals with a one-to-one non-symmetric matching model, described in the context of a job market. It is a special case of the model of Eriksson and Karlander. The restriction, plausible for a great variety of markets, is that **all workers are flexible while some firms can transfer utility and others cannot.**

This market can be viewed as resulting from the entrance of the rigid firms into the Assignment Market or from the entrance of the flexible firms into the Marriage Market. The investigation of the effects caused by such an event on the agents' payoffs is of economic interest, by the extent that these effects capture fundamental differences and similarities between the two types of markets. The main issue of such a study is to determine which points, x in the core of the hybrid market, and y in the core of the non-hybrid market, should be compared. These points must be such that the distinction between them expresses the effects mentioned above. Intuitively, it should be required that after the entrance of the new firms, the agents who were in the non-hybrid market can get outcome x, by continuing to do the same sort of things they were doing to reach outcome y. That is, it is as if the two outcomes were obtained via the application of the same allocation rule.

For the non-hybrid markets, the outcomes commonly picked to study comparative static effects are the extreme points of the lattice of stable payoffs. This is because they always exist and can be obtained via some well known algorithms, which can be reproduced after the addition of the new agents.[6]

The first step in the investigation of these effects shows that results (A), (B) and (C) are not robust to the introduction of flexible (respectively, rigid) firms into the Marriage (respectively, Assignment) market. Consequently, the optimal stable outcomes for each side not always exist.

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[5] We will call the Marriage model and the Assignment game non-hybrid models.

[6] See Gale and Shapley (1962) for an algorithm applied to the Marriage Market and Demange (1982), Leonard (1983) and Demange, Gale and Sotomayor (1986), for an algorithm applied to the flexible market.

The characterization of the core of the hybrid market without imposing any condition, and so that new representative points in the cores can be used appropriately to investigate the specific comparative static questions posed above, is then the theme of the rest of the present paper.

Our main finding is that the core of the hybrid model can be characterized by a convenient partition whose elements are complete lattices, endowed with properties (B) and (C). Furthermore, this new algebraic structure reflects a sort of robustness to the exit of flexible (respectively, rigid) firms from the hybrid market in the following sense. As all flexible (respectively, rigid) firms leave the market, the number of lattices in the partition decreases to one. Thus, the resulting algebraic structure coincides with the complete lattice structure of the core of the Marriage model (respectively, Assignment model). Therefore, the extreme points of the lattices of the core partition coincide with the respective extreme points of the lattices of the non-hybrid models. We call these core payoffs "quasi-optimal stable payoff for firms" and "quasi-optimal stable payoff for workers". There may be several quasi-optimal stable payoffs for firms and several quasi-optimal stable payoffs for workers, whose number depends on the number of subsets of the core partition of the hybrid market.

The important implication of the robustness property mentioned above is that, if we select, say, any of the quasi-optimal stable payoff for firms, the distinction between this outcome and the firm-optimal stable payoff for the Marriage (respectively, Assignment) model expresses the effects, on the selected outcome, caused by the exit of all flexible (respectively, rigid) firms from the hybrid market. Thus, the non-existence of the optimal core outcomes for each side of the hybrid model is not a hindrance to get meaningful comparative static results. Our theorems imply that, when all flexible (respectively, rigid) firms exit the hybrid market, all firms (weakly) gain and all workers (weakly) lose if a quasi-optimal stable payoff for one side of the market always prevails.

Properties (A), (B) and (C) are characteristic of the core of a wide variety of non-hybrid two-sided matching markets. It is worth to point out that the fact that these properties persist, in a consistent way, for the core partition of the hybrid market, indicates that they may be even more fundamental than previous results have suggested.

Each subset of the core partition of the hybrid model is obtained as follows. For any stable matching[7] x, decompose the market participants into two subsets. One subset contains all rigid firms and their mates at x and the other one contains all flexible firms, their mates at x and the unmatched workers. Now, fix such agents' partition. The desired subset is formed with the core outcomes under which firms are matched to workers within each subset of the agents' partition. Roughly speaking, we can then imagine each subset of the core partition as if it was the juxtaposition of a subset of the core of a Shapley-Shubik market and a subset of the core of a Gale-Shapley market. The standard arguments for properties (B) and (C) then go through within each market. Therefore, these results are obtained for each element of the core partition. For the lattice result the conclusion is not so straightforward.

Clearly, this core partition presents the robustness property mentioned above, since as the rigid firms exit the hybrid market, the core partition is reduced to only one set, since any stable matching is compatible with any core payoff (Shapley and Shubik, 1972). Then, the extreme points of this set are the corresponding extreme points of the lattice of the core of the resulting Assignment market. An analogous result holds as the flexible firms disappear, due to the fact that the matched agents in the Marriage market are the same at every stable matching, which is implied by property (C).

Of course, there exist other core partitions whose elements satisfy properties (A), (B) and (C). Consider, for instance, the partition defined by the stable matchings. For every fixed stable matching x take all stable outcomes compatible with x. However, when the flexible firms exit the hybrid market, each stable matching for the resulting Marriage market is an element of the core partition. Thus, the algebraic structure of the core of the Marriage model, which is induced by that of the core of the hybrid model, is not the standard one. However, although every stable matching x is both supremum and infimum of the lattice $\{x\}$, an arbitrary stable matching is not necessarily a firm-optimal or a worker-optimal stable matching. On the other hand, the effects on the extreme points of the lattices of the core partition of the hybrid market, caused by the exit of the flexible firms, are not always consistent with the differences between these extreme outcomes and an arbitrary stable matching. Therefore, the

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[7] The matching x is stable if it is compatible with some stable payoff.

partition of the core defined by the stable matchings does not provide the desired characterization.

A special case of the hybrid market studied here is obtained when rigid firms do not discriminate workers. In many countries, this is the case of job markets in the entrance level, in which firms are public institutions. To fulfill the vacancies of a given position they offer the same payment to all vacancies. The point is that, if rigid firms offer workers equal salary for the same position, all workers will have the same list of preference over rigid firms. (Observe that this does not imply that workers are indifferent over rigid firms). The introduction of this particularity in the model causes the set of stable outcomes of the rigid market to have only one element. How does this particularity affect our results? Clearly, the fact that every worker has the same list of preference over the rigid firms causes the set of stable outcomes of the rigid market to be a singleton. There could be other conditions under which the core of a non-hybrid market is a singleton.[8] For the cases where the core of some of the non-hybrid markets has only one element, we present a more general result. Namely, *when the core of the Marriage (respectively, Assignment) market is a singleton, every worker weakly prefers to trade in the hybrid market and every rigid (respectively, flexible) firm weakly prefers to trade in the rigid (respectively, flexible) market, under any arbitrary stable allocation procedure.*

As for the general problem of obtaining meaningful comparative static results of adding agents from the same side to the hybrid market, it is easy to construct examples where the comparison between two quasi-optimal stable outcomes for the same side of the market may be meaningless. Thus, the algebraic structure of the core of the hybrid market presented in this paper is not useful to approach such a general question. Although this work is not primarily concerned with investigating general comparative static questions for the hybrid market, we were able to prove, in this direction, a partial result that can be a starting point for future investigations: *When the players are*

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[8] Eeckhout (2000) proposes a necessary and sufficient condition on agents' preferences, under which the core of a matching market is a singleton. Clark (2003) proposes a sufficient condition (called the No Crossing Condition), which is closely related to the well-known Single Crossing Condition. Ehlers and Massó (2004) relates singleton cores to incentive-compatibility in an incomplete information setup. Sonmez (1999) showed for general allocation problems with indivisibilities that a mechanism is incentive-compatible, Pareto-optimal and individually rational if and only if for each problem the core is a singleton and the mechanism chooses this allocation.

allocated according to some quasi-optimal stable outcome for workers (respectively, firms) x, *and one or more firms (respectively, workers) are added to the pool of the hybrid market, then there is some quasi-optimal stable outcome for workers (respectively, firms)* y *in the core of the new market, under which no worker (respectively, firm) is worse off and no flexible firm (respectively, no worker matched to a flexible firm) is better off. In addition, if* x *does not have any weak blocking pair then no rigid firm (respectively, worker) is better off.* This result is obtained via the theory of *simple outcomes*[9] applied in Sotomayor (2000) to show the existence of stable outcomes for the hybrid model.

This paper is organized as follows. Section 2 presents the mathematical model. Section 3 is devoted to the structural properties of the core. The subsequent section presents the main results. Section 5 provides the conceptual framework of simple outcomes and proves some comparative static result of adding agents from the same side to the hybrid market. Section 6 presents our final remarks and discusses some new directions and related works.

1.2 UNDERLYING ECONOMIC ASSUMPTIONS

The assumptions of our model are those of Gale and Shapley for the Marriage Market and of Shapley and Shubik for the Assignment Game. Albeit restrictive in many aspects, they allow us to include a wide variety of two-sided markets, such as housing markets, used car markets, and labor markets in general. The major economic hypothesis is: **Utility is identified with money.**

It is true that the preferences of workers rely not only on salary, but also on other factors such as satisfaction with their jobs, location of the company, etc. On the other hand, most workers who change jobs deliberately justify that they do so when they have the chance to earn better salaries. This means that in a great number of situations the effect that income produces on people's choices renders the effect of other factors negligible, in such a way that the complete monetarization of the utility is defensible. Thus, it is reasonable to think of an open market with several workers selling their services for money and several firms buying such services.

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[9] A feasible outcome is simple for one of the sides if no matched agent of this side forms a blocking pair.

2. THE FRAMEWORK

The model described here is a special case of the one due to Eriksson and Karlander (2000), which was inspired in that of Roth and Sotomayor (1996). There are two finite and disjoint sets of agents, $P = \{p_1, p_2, ..., p_i, ... p_m\}$ and $Q = \{q_0, q_1, q_2, ..., q_j, ..., q_{n-1}\}$, where q_0 is a fictitious worker, included for technical reasons (we could have included a fictitious firm, instead). The elements of P will be named firms and those of Q will be referred to as workers. The agents on one side form partnerships with the agents on the opposite side. Each agent is interested in forming one partnership at most. Some firms are *rigid* for sharing the gains of their partnerships while others are *flexible.*

If firm p_i is *rigid* and forms a partnership with worker q_j, it determines the salary of its partner: p_i receives $u_i=a_{ij}$ and q_j receives $v_j=b_{ij}$, with $a_{i0}=0$ and $b_{i0}=0$. If p_i is not rigid we will say it is *flexible.* If flexible firm p_i and worker q_j form a partnership, the gain of trade is c_{ij}. Agent p_i receives u_i and q_j receives v_j, such that $u_i + v_j=c_{ij}$. We will assume that $c_{ij}\geq 0$. If $q_j=q_0$ then $c_{i0}=0$, so $u_i=v_j=0$. We will denote the set of rigid firms and the set of flexible firms by P^R and P^F, respectively. For simplicity, we will consider that the reservation payoff of each agent is 0.

Following the usual assumptions for the Marriage Market, which warrant the existence of the optimal stable matchings for each side of the market, we will consider that preferences of rigid firms over workers, as well as preferences of workers over rigid firms, are strict, complete and transitive.

For a rigid firm the a_{ij}'s express its preferences over workers. Thus, numbers a_{ij}'s corresponding to rigid firm p_i are distinct; p_i prefers q_j to q_k if and only if $a_{ij}>a_{ik}$, and q_j is **acceptable** to p_i if and only if $q_j=q_0$ or $a_{ij}>0$. This way, each rigid firm can list the workers it would like to hire, in a strict order of preference. If, for instance, the list of preferences of p_i is given by $L(p_i) = q_h, q_j, \ldots, q_0, q_p,$ then it means that p_i prefers q_h to q_j. Worker q_p is unacceptable to p_i, in the sense that p_i prefers to hire nobody to hiring q_p.

Given the fact that workers have strict, complete, and transitive preferences over rigid firms, they can also express their preferences over these agents in a strict order. Worker q_j prefers rigid firm p_i to rigid firm p_h if and only if $b_{ij}>b_{hj}$. Firm $p_i\in P^R$ is acceptable to worker q_j if $b_{ij}>0$.

The hybrid market is then given by **(P,Q,a,b,c),** where **a** is the matrix of a_{ij}'s, **b** is the matrix of b_{ij}'s and **c** is the matrix of c_{ij}'s. The market **(P^F,Q,c)** will be called **flexible market,** whereas **(P^R,Q,a,b)** will be called **rigid market.**

Informally, a *matching* is an **allocation of the firms to the workers.** A matching will be denoted by x (The formal definition may be seen in Roth and Sotomayor, 1990, where an overview of the discrete and continuous one-to-one matching models is presented). It is **feasible** if every firm is allocated to one worker (who might be the fictitious worker) and if no real worker is assigned to more than one firm. If worker q_j is assigned to firm p_i by x then we may write some times $q_j=x(p_i)$ or $p_i=x(q_j)$. If p_i is matched to the fictitious worker then we will say, some times, that p_i is **unmatched**. If q_j is not allocated to any firm we will say that he/she is **unmatched** and will write $x(q_j)=q_j$. Worker q_0 can be assigned to any number of firms. Given a set S of agents belonging to the same side of the market we denote **$x(S)\equiv\{x(s);\ s\in S\}$.**

An **outcome** is a matching x plus a pair of vectors (u,v) called payoff, with $u\in R^m$ and $v\in R^n$. It will be denoted by (u,v;x).

Definition 1. *An outcome (u,v;x) is **feasible** if x is feasible and*

a) $u_i\geq 0$ *and* $v_j\geq 0$ *for every* $(p_i,q_j)\in PxQ$ *(individual rationality);*

b) $u_i=0$ *if* p_i *is matched to* q_0 *under* x;

c) $v_j=0$ *if* q_j *is unmatched under* x;

d) $u_i+v_j=c_{ij}$ *if* q_j *is matched to* $p_i\in P^F$ *by* x;

e) $u_i=a_{ij}>0$ *and* $v_j=b_{ij}>0$ *if* $q_j\neq q_0$ *is matched to* p_i *by* x *and* $p_i\in P^R$.

If (u,v;x) is feasible then (u,v) is a feasible payoff and we say that (u,v) is **compatible with x** or **x is compatible with (u,v).**

The solution concept for this sort of game is that of stability.

Definition 2. *An outcome (u,v;x) is **stable** if it is feasible and for every* $(p_i,q_j)\in PxQ$ *we have*

a) $u_i+v_j\geq c_{ij}$ *if* $p_i\in P^F$ *and*

b) $u_i>a_{ij}$ *or* $v_j\geq b_{ij}$ *if* $p_i\in P^R$.

If a) or b) is not satisfied for some pair (p_i,q_j), we say that this pair **blocks** the outcome. Thus, **an outcome is stable if it is feasible and does not have any blocking pair.**

If (u,v;x) is stable we say that (u,v) is a **stable payoff** and **x is a stable matching.** Then, a stable matching is a matching which is compatible with a stable payoff.

Since the pairs (p_i,q_j)'s are the only essential coalitions, an outcome is stable if and only if it is in the core of the cooperative game induced by market (P,Q,a,b).

Definition 3. *The payoff (u,v) is a* ***firm-optimal stable payoff*** *(or* ***optimal stable payoff for the firms)*** *if* $u_i \geq u'_i$ *for every stable payoff (u',v'). Symmetrically, (u,v) is a* ***worker-optimal stable payoff*** *(or* ***optimal stable payoff for the workers)*** *if* $v_j \geq v'_j$ *for every stable payoff (u',v').*

If (u,v) is an optimal stable payoff for firms (respectively, workers) and x is compatible with (u,v), then (u,v;x) is an optimal stable outcome for firms (respectively, workers).

2.1 THE RIGID MARKET: (P^R,Q,a,b)

All agents may list in order of strict preference their potential partners. This market is the *Marriage Market*, introduced by Gale and Shapley in 1962. The outcome **(u,v;x) is stable if it is feasible and no pair (p_i, q_j) exists such that q_j prefers p_i to $x(q_j)$ and p_i prefers q_j to $x(p_i)$.** The existence theorem for the Marriage market is proved by Gale and Shapley (1962) and Sotomayor (1996).

The existence of the optimal stable outcomes for the Marriage market was proved by Gale and Shapley and is also implied by the lattice structure of the core, whose proof is attributed to Conway by Knuth (1976). (See also Gale and Sotomayor, 1985-a).

2.2 THE FLEXIBLE MARKET: (P^F,Q,c)

According to our previous assumption, $c_{ij} \geq 0$ for every $(p_i,q_j) \in P^F xQ$ and the reservation payoff of each agent is 0. This is the *Assignment Game* introduced by Shapley and Shubik (1972).

An outcome (u,v;x) is feasible if it satisfies a), b), c) and d) in Definition 1 and is stable if it is feasible and satisfies a) in Definition 2. **Therefore, if (u,v;x) is stable, then each firm p_i is maximizing its utility payoff at salaries v, every worker q_j is maximizing his utility payoff at payoffs u and every unmatched agent has payoff 0.** The existence theorem for the Assignment game is proved by Shapley and Shubik (1972) and Sotomayor (2000).

Analogously to the rigid market, and due to Shapley and Shubik, there is always one and only one optimal stable payoff for firms and one and only one optimal stable payoff for workers.

3. STRUCTURAL PROPERTIES OF THE CORE

3.1 DISSIMILARITIES BETWEEN THE CORES OF THE HYBRID AND OF THE NON-HYBRID MARKETS

This sub-section shows that properties (A), (B) and (C) presented and discussed in session 1, that characterize the core of the Marriage and of the Assignment models, may fail to hold in the hybrid model, **even under the stronger assumption that the preferences of the workers over the rigid firms are the same for every worker**.

Example 1. (Lattice property fails to hold; there are no firm-optimal and no worker-optimal stable payoff) Consider $P^R=\{p_1,p_2,p_3\}$, $P^F=\{p_4\}$ and $Q=\{q_1,q_2,q_3,q_4,q_0\}$. The vector of c_{4j}'s, j=1,2,3,4 is (1.5, 4, 5, 0); the lists of preference of the rigid firms are given by: $L(p_1)= q_3, q_4, q_0$; $L(p_2)= q_2, q_1, q_0$; $L(p_3)= q_1, q_0$. The numbers b_{ij}'s are: $b_{1j}=4$, $b_{2j}=3$, $b_{3j}=2$ for all j=1,2,3,4.

It is a matter of verification that there only exist two stable outcomes for this market. The first one is compatible with matching x, where $x(p_1)=q_3$, $x(p_2)=q_1$, $x(p_3)=q_0$, $x(p_4)=q_2$. The payoff vector for workers is v=(3,3,4,0,0). The second one is compatible with matching x', where $x'(p_1)=q_4$, $x'(p_2)=q_2$, $x'(p_3)=q_1$, $x'(p_4)=q_3$. The payoff vector for workers is v'=(2,3,4,4,0).

There is no stable outcome that is optimal for firms: p_1 prefers the first outcome while p_2 and p_3 prefer the second one. It is also true that **there is no stable payoff that is optimal for workers**: q_1 prefers de first outcome while q_4 prefers the second one. Clearly, **the set of stable payoffs is not a lattice**, neither under the partial order defined by the preferences of the workers, nor under the partial order defined by the preferences of the firms.■

Example 2. (Polarization of interests fails to hold). Consider $P^R=\{p_1\}$, $P^F=\{p_2,p_3\}$ and $Q=\{q_1,q_2,q_0\}$; $c_{21}=3$, $c_{22}=0$; $c_{31}=2$, $c_{32}=5$; $b_{1j}=3$ for all $j=1,2$; the list of preference of rigid firm p_1 is given by: $L(p_1)= q_1, q_0$. The outcomes $(u,v;x)$ and $(u',v';x')$ are stable, where $x(p_1)=q_1$, $x(p_2)=q_0$, $x(p_3)=q_2$; $u_2 = u_3 =0$; $v_1 =3$, $v_2 = 5$ and $x'(p_1)=q_0$, $x'(p_2)=q_1$, $x'(p_3)=q_2$; $u'_2 = 0$, $u'_3 = 2$; $v'_1 = 3$, $v'_2 = 3$.

There is no polarization of interests between firm p_1 and workers, with respect to the outcomes $(u,v;x)$ and $(u',v';x')$, since the first outcome is better than the second one for firm p_1 and workers weakly prefer the first outcome to the second one.■

Example 3. (An unmatched agent may receive a positive payoff under a different outcome) Consider Example 1 again. It can be seen that the rigid firm p_3 is unmatched under x and matched under x', where $x(p_1)=q_3$, $x(p_2)=q_1$, $x(p_3)=q_0$, $x(p_4)=q_2$ and $x'(p_1)=q_4$, $x'(p_2)=q_2$, $x'(p_3)=q_1$, $x'(p_4)=q_3$. Also worker q_4 is unmatched under x and has payoff $v'_4=4$ in the other outcome. ■

3.2 CHARACTERIZATION OF THE CORE OF THE HYBRID MARKET

This section shows that the core of the hybrid market is endowed with an algebraic structure, whose restriction to the non-hybrid markets coincides with the complete lattice structure of the core of these markets. It consists of a finite and disjoint union of complete lattices that have properties (A), (B) and (C). Consequently, each of these subsets of the core presents, in this extent, the same similarities as the non-hybrid cores. This characterization will be used in the next section to derive meaningful comparative static results.

Let x be a stable matching for the market $M=(P,Q,a,b,c)$. Define,

$C(x) \equiv \{(u,v);\ (u,v;x') \text{ is stable for some } x' \text{ and } x(P^R)=x'(P^R)\}$.

Of course, given the stable matchings x and x', $C(x)=C(x')$ if and only if $x(P^R)=x'(P^R)$. Also, the core of M is the disjoint union of the distinct sets $C(x)$'s.

Property 1 asserts that there is a polarization of interests between firms and workers along sets $C(x)$'s.

Property 1. *Let $(u,v;x)$ and $(u',v';x')$ be stable outcomes in $C(x)$. Then $u_i \geq u'_i$ for all $p_i \in P$ if and only if $v_j' \geq v_j$ for all $q_j \in Q$.*

Proof. Suppose $u_i \geq u'_i$ for all $p_i \in P$. Suppose by way of contradiction $v_j' < v_j$ for some $q_j \in Q$. Then, there is some $p_i \in P$ such that $x(q_j)=p_i$. In this case, $p_i \in P^F$ for otherwise we would have $u'_i=u_i$, due to stability of (u',v'). But this is not possible because the preferences are strict and $x(p_i) \neq x'(p_i)$, contradiction. Therefore, $c_{ij}=u_i + v_j > u'_i + v'_j$, which contradicts stability of $(u',v';x')$. Hence, $v_j \leq v'_j$ for all $q_j \in Q$.

In the other direction, suppose $v_j \leq v'_j$ for all $q_j \in Q$. Suppose by way of contradiction $u_i < u'_i$ for some $p_i \in P$. Then, there is some $q_j \in Q$ such that $x'(q_j)=p_i$. In this case $p_i \in P^F$, for otherwise we would have $x(p_i) \neq x'(p_i)=q_j$ on one hand and $v'_j=v_j$ on the other hand, due to the stability of (u,v). Nevertheless, $x(P^R)=x'(P^R)$ implies that $x(q_j) \in P^R$, so q_j is matched under x to a rigid player other than p_i, so $v'_j \neq v_j$ due to the strictness of the preferences over the rigid firms, which is absurd. Therefore, $c_{ij}=u'_i + v'_j > u_i + v_j$, which contradicts stability of $(u,v;x)$. Hence, $u_i \geq u'_i$ for all $p_i \in P$. ■

Lemma 1. *Let $\sigma=(u,v;x)$ and $\sigma'=(u',v';x')$ be stable outcomes in $C(x)$. Set $P(\sigma) \equiv \{p_i \in P;\ u_i > u'_i\}$ and $Q(\sigma') \equiv \{q_j \in Q;\ v'_j > v_j\}$. Then, x' and x match every worker of $Q(\sigma')$ to a firm in $P(\sigma)$ and every firm of $P(\sigma)$ to a worker in $Q(\sigma')$.*

Proof. It suffices to show that $x(P(\sigma)) \subseteq Q(\sigma')$, $q_0 \notin Q(\sigma')$, $x'(Q(\sigma')) \subseteq P(\sigma)$ and then to use that x and x' are one-to-one and $P(\sigma)$ and $Q(\sigma')$ are finite.

To prove that $x(P(\sigma)) \subseteq Q(\sigma')$ take $p_i \in P(\sigma)$. Then, $u_i > u'_i \geq 0$ and so p_i is designated by x to some non-fictitious worker. Let $x(p_i)=q_j$. If $p_i \in P^F$ then $v_j < v'_j$ for otherwise $c_{ij}=u_i + v_j > u'_i + v'_j$, which contradicts the stability of $(u',v';x')$. If $p_i \in P^R$ then $x'(p_i) \neq q_j$, due to the strictness of the preferences of p_i, and $v_j = b_{ij} > 0$. We claim that

$v'_j>v_j$. In fact, since $x(P^R)=x'(P^R)$ it follows that $x'(q_j)\in P^R$, so $v'_j\neq v_j$, by the strictness of the preferences over the rigid firms. However, if $v'_j<v_j$ then (p_i,q_j) would block (u',v'), contradiction. This proves that $q_j \in Q(\sigma')$. Hence, $x(P(\sigma))\subseteq Q(\sigma')$.

To see that $x'(Q(\sigma'))\subseteq P(\sigma)$ take $q_j\in Q(\sigma')$. Then, $v'_j>v_j \geq 0$, so q_j is not q_0 and is matched at x' to some p_i. We claim that $u_i>u'_i$. In fact, suppose by way of contradiction $u_i\leq u'_i$. If $p_i\in P^R$, then the fact that $v'_j>v_j$ implies that $x(q_j)\neq p_i$, and the stability of (u,v) implies that $u_i\geq u'_i$, so $u_i=u'_i$. But then the strictness of the preferences of the rigid firms implies that $x(p_i)=x'(p_i)=q_j$, which is a contradiction. Thus, if $u_i\leq u'_i$ then $p_i\in P^F$ and so $c_{ij}= u'_i+v'_j > u_i + v_j$, which contradicts the stability of (u,v). This way, $u_i>u'_i$, so $p_i\in P(\sigma)$. Hence, $x'(Q(\sigma'))\subseteq P(\sigma)$ and the proof is complete. ■

Property 2. *Let $\sigma=(u,v;x)$ and $\sigma'=(u',v';x')$ be stable outcomes in $C(x)$. If p_i (respectively q_j) is unmatched under x', then $u_i=0$ (respectively $v_j=0$).*

Proof. Define $P(\sigma)$ as in Lemma 1. Suppose p_i is unmatched under x'. If $u_i>0=u'_i$ then p_i is in $P(\sigma)$, so Lemma 1 implies p_i is matched under x', which is a contradiction. With a symmetric argument for q_j we complete the proof. ■

Define the partial order $\geq_P$ in $C(x)$ as follows. Let (u,v) and (u',v') be stable payoffs in $C(x)$. Then, $(u,v) \geq_P (u',v')$ if and only if $u_i\geq u'_i$ for all $p_i\in P$. Symmetrically, we define the partial order $\geq_Q$ in $C(x)$. Property 1 implies that $(u,v) \geq_P (u',v')$ if and only if $(u',v') \geq_Q (u,v)$. Hence, these partial orders are well defined (satisfy de anti-symmetric property).

Notice that the restriction of $C(x)$ to the flexible firms and to the workers who are not matched to the rigid firms is a subset (and probably a proper subset) of the core of the assignment game involving only these agents. However, the fact that the core of the assignment game is a lattice does not allow us to conclude that $C(x)$ is a lattice. This fact is not so straightforward and we will need two more lemmas to prove it.

Lemma 2. *With the same hypotheses and notations of Lemma 1 we have that* *(a)* $(u\vee u',v\wedge v';y)$ *is stable for* M, *where* $(u\vee u')_i=\max\{u_i, u'_i\}$ *and* $(v\wedge v')_j=\min\{v_j,v'_j\}$ *for all* $p_i\in P$ *and* $q_j\in Q$, *and* y *agrees with* x *on* $P(\sigma)\cup Q(\sigma')$ *and with* x' *on the other agents.*

(b) $(u\wedge u',v\vee v';y')$ *is stable for* M, *where* $(u\wedge u')_i=\min\{u_i, u'_i\}$, *and* $(v\vee v')_j=\max\{v_j,v'_j\}$ *for all* $p_i\in P$ *and* $q_j\in Q$, *and* y' *agrees with* x' *on* $P(\sigma)\cup Q(\sigma')$ *and with* x *on the other agents.*

Proof. We will prove part (a). With symmetric arguments we can prove part (b). Then, for every $(p_i,q_j)\in PxQ$, denote $u^+_i\equiv(u\vee u')_i$ and $v^-_j\equiv(v\wedge v')_j$. We must show that $(u^+,v^-;y)$ is stable for M. To see that $(u^+,v^-;y)$ is feasible, first observe that Lemma 1 implies that y is a feasible matching. Conditions a), b) and c) of Definition 1 follow from the feasibility of (u,v;x) and (u',v';x'). Therefore, it remains to show that if $y(p_i)=q_j$ then d) $u^+_i+v^-_j=c_{ij}$ if $p_i\in P^F$ and e) $u^+_i=a_{ij}>0$, $v^-_j=b_{ij}>0$ if $p_i\in P^R$. Part d) and e) are immediate by Lemma 1 if $p_i\in P(\sigma)$. If $p_i\notin P(\sigma)$ then $q_j=y(p_i)=x'(p_i)$, $u'_i\geq u_i$, so $u^+_i=u'_i$. On the other hand, Lemma 1 implies that $q_j\notin Q(\sigma')$, so $v_j\geq v'_j$ and so $v^-_j=v'_j$. Hence, $u^+_i+v^-_j=u'_i+v'_j=c_{ij}$ if $p_i\in P^F$. If $p_i\in P^R$ then, $v^-_j=v'_j=b_{ij}>0$ and $u^+_j=u'_i=a_{ij}>0$. Therefore, $(u^+,v^-;y)$ is feasible.

That $(p_i,q_j)\in PxQ$ does not form a blocking pair is immediate if $p_i\in P^F$ and $u^+_i=u_i$ and $v^-_j=v_j$ or $u^+_i=u'_i$ and $v^-_j=v'_j$ or if $p_i\in P^R$. In this last case, if $v^-_j=v_j<b_{ij}$ then $u_i\geq a_{ij}$, by stability of (u,v;x), so $u^+_j\geq u_i\geq a_{ij}$ and so (p_i,q_j) cannot be a blocking pair; if $v^-_j=v'_j<b_{ij}$ then $u'_i\geq a_{ij}$, by stability of (u',v';x'), so $u^+_j\geq u'_i\geq a_{ij}$ and so (p_i,q_j) cannot be a blocking pair. For the other cases, if $p_i\in P^F$, $u^+_i=u_i$ and $v^-_j=v'_j$, then $u^+_i+v^-_j=u_i+v'_j\geq u'_i+v'_j\geq c_{ij}$ by stability of (u',v';x'); if $p_i\in P^F$, $u^+_i=u'_i$ and $v^-_j=v_j$, then $u^+_i+v^-_j=u'_i+v_j\geq u_i+v_j\geq c_{ij}$ by stability of (u,v;x). Therefore $(u^+,v^-;y)$ is stable for M. ■

Property 3. *If* z *is a stable matching for* $M=(P,Q,a,b,c)$, *then* $C(z)$ *is a non-empty and complete lattice under* $\geq_P$ *and* $\geq_Q$.

Proof. Since z is compatible with some stable payoff and this stable payoff is in C(z), it follows that C(z) is non-empty. If C(z) is a singleton, then we are done. Otherwise, take (u,v) and (u',v') in C(z), compatible with matchings x and x', respectively. Set $\sigma\equiv(u,v;x)$ and $\sigma'\equiv(u',v';x')$ and define $P(\sigma)$ and $Q(\sigma')$ as in Lemma 1. We have that $x(P^R)=x'(P^R)=z(P^R)\equiv A$. From Lemma 2, $(u\vee u',v\wedge v';y)$ and $(u\wedge u',v\vee v';y')$

are stable for M, where y matches all of $Q(\sigma')\cup P(\sigma)$ according to x and the other agents according to x'; y' matches all of $Q(\sigma')\cup P(\sigma)$ according to x' and the other agents according to x. Then, if $p_i\in P^R$ we have that $y(p_i)=x(p_i)$ or $y(p_i)=x'(p_i)$. In any case $y(p_i)\in A$, so $y(P^R)\subseteq A$. Since $|y(P^R)| =|P^R|=|A|$ it follows that $y(P^R)=A$. Analogously, $y'(P^R)=A$. Hence, $(u \vee u',v\wedge v')$ and $(u \wedge u',v\vee v')$ are in $C(z)$, so the supremum and the infimum of (u,v) and (u',v') under both partial orders, $\geq_P$ and $\geq_Q$, exist. This proves that $C(z)$ is a lattice under $\geq_P$ and $\geq_Q$. It is not hard to show that $C(z)$ is a compact subset of R^{n+m}, since it is the set of solutions of a linear system of non-strict inequalities and equalities, and every individual payoff is bounded. Hence, $C(z)$ is a complete lattice and the proof is complete. ■

Definition 4. *Let x be a stable matching for (P,Q,a,b,c). Let $(u,v)\in C(x)$. The payoff (u,v) is a* ***quasi-optimal stable payoff for firms (or quasi-firm-optimal stable payoff) in $C(x)$*** *if it is stable and $u_i\geq u'_i$ for all $p_i\in P$ and every stable payoff $(u',v')\in C(x)$. We say that (u,v) is* ***quasi-optimal stable payoff for firms*** *if it is quasi-optimal stable payoff for firms in $C(x)$, for some stable matching x. Symmetrically, we define a* ***quasi-optimal stable payoff for workers (or quasi-worker-optimal stable payoff) in $C(x)$ and*** *a* ***quasi-optimal stable payoff for workers.***

That is, a quasi-optimal stable payoff for firms (respectively, workers) is the supremum of some of the lattices $C(x)$, under the partial order defined by the preference of the firms (respectively, workers).

It follows from Definition 4 and Property 3 that the set of extreme points of the lattices $C(x)$'s is exactly the union of the set of quasi-optimal stable payoffs for firms and the set of quasi-optimal stable payoffs for workers.

Property 1 implies that (u,v) is a quasi-firm-optimal stable payoff in $C(x)$ if and only if $(u,v)\geq_P(u',v')$ and $(u',v')\geq_Q(u,v)$, for every stable payoff $(u',v')\in C(x)$. Symmetrically, (u,v) is a quasi-worker-optimal stable payoff in $C(x)$ if and only if $(u,v)\geq_Q(u',v')$ and $(u',v')\geq_P(u,v)$, for every stable payoff $(u',v')\in C(x)$.

Corollary 1. *There always exist at least one quasi-optimal stable payoff for firms and one quasi-optimal stable payoff for workers in market (P,Q,a,b,c).*

Proof. The set of stable matchings of (P,Q,a,b,c) is non-empty, since the set of stable outcomes for this market is non-empty (Theorem 1 of Sotomayor, 2000). Then, choose a stable matching z. By Property 3, C(z) is a complete lattice under $\geq_P$ and $\geq_Q$, whose extreme points are the desired outcomes. ■

If M is the rigid market then C(x) is the set of all stable payoffs, since the set of matched agents under x is the same as under any other stable matching. This result is obviously true if M is the flexible market. Then, the quasi-optimal stable payoff for firms (respectively, workers) is the optimal stable payoff for firms (respectively, workers) in the non-hybrid markets.

It must be pointed out that a quasi-optimal stable payoff for firms in the hybrid market, for example, need not to be an optimal stable payoff for firms in case this outcome exists. See the following example.

Example 4. **(a quasi-firm-optimal stable payoff that is not firm-optimal)** Consider $P^R=\{p_1\}$, $P^F=\{p_2\}$, $Q=\{q_1,q_2,q_0\}$. Let $b_{11}=b_{12}=1$, $c_{21}=c_{22}=3$ and let $L(p_1)=q_2,q_1,q_0$. We have that $(u^1,v^1;x^1)$ is the firm-optimal stable outcome, where $x^1(p_1)=q_2$, $x^1(p_2)=q_1$, $u^1_2=3$, $v^1_1=0$ and $v^1_2=1$. However $(u^2,v^2;x^2)$ is also a quasi-optimal stable outcome for firms, where $x^2(p_1)=q_1$, $x^2(p_2)=q_2$, $u^2_2=2$, $v^2_1=v^2_2=1$.ν

Remark 1. There are other partitions of the core whose elements also satisfy properties (A), (B) and (C). Let x be a stable matching for the hybrid market M. Let C'(x) denote the set of stable payoffs for M, (u,v), such that x is compatible with (u,v). Clearly, the core of the hybrid market is the union of the sets C'(x)'s and $C'(x)\subseteq C(x)$ for every stable matching x. With analogous arguments to those used to prove the results of this section, it can be shown that C'(x) is a complete lattice which satisfies properties (B) and (C). However, when the flexible firms exit the hybrid market, the algebraic structure of the core of the Marriage model induced by that of the core of the hybrid model is not the standard one. The partition of the core is formed by singleton sub sets. Clearly, any stable matching *x* is both, maximal and minimal element of the lattice {x}, but it is not, necessarily, the maximal or minimal element of the lattice of the core. Example 5 below illustrates a situation in which the differences between these outcomes and an arbitrary stable matching not always may be attributed to the entrance of flexible firms in the rigid market.

Example 5. Consider $P^R=\{p_1,p_2\}$, $P^F=\{p_3\}$, $Q=\{q_1,q_2,q_3,q_0\}$; $b_{11}=1$, $b_{12}=2$, $b_{21}=2$, $b_{22}=1$, b_{13} and b_{23} may be arbitrary; $c_{31}=1$, $c_{32}=1$, $c_{33}=3$. The lists of preference of the rigid firms are given by: $L(p_1)= q_1, q_2, q_3, q_0$; $L(p_2)= q_2, q_1, q_3, q_0$.

From the numbers b_{ij}'s we can deduce that the lists of preference of workers q_1 and q_2 over the rigid firms are given by: $L(q_1)= p_2, p_1$; $L(q_2)= p_1, p_2$.

The worker-optimal stable outcome for the rigid market is (u',v';x'), where $x'(p_1)=q_2$, $x'(p_2)=q_1$, $v'_1=v'_2=2$, $v'_3=0$. Observe that q_3 is unmatched. If p_3 enters the market, the outcome (u*,v*;x*), where the new firm matches with the unemployed worker and pays him 3, but the other workers and rigid firms do not change their mates under the original matching x', is the most preferable stable outcome for workers in C'(x*). The entry of the new firm makes no worker worse off and no rigid firm better off under (u*,v*;x*) than under (u',v';x').

Now consider the outcome (u,v;x), where $x(p_1)=q_1$, $x(p_2)=q_2$, $x(p_3)=q_3$, $v_1=v_2=1$, $v_3=3$, and $u_3=0$. Observe that workers q_1 and q_2 are re-matched as they had been in the firm-optimal stable outcome, instead of in the worker-optimal stable outcome. Rigid firms prefer (u,v;x) to (u',v';x'), while workers q_1 and q_2 prefer the opposite. Clearly, the distinctions between (u',v';x') and (u,v;x) cannot be attributed to the entrance of p_3 in the rigid market. However, (u,v;x) is the most preferable stable outcome for workers in C'(x). ■

4. MAIN RESULTS

This section studies the effects on agents' payoffs caused by the entrance of rigid firms into the flexible market and of flexible firms into the rigid market, according to some specific stable allocation procedure.

Suppose workers are allocated to firms according to an allocation procedure H, which produces, for each market M (hybrid, rigid or flexible), an outcome H(M) for market M. If H(M) is always stable with respect to M, it will be called a stable allocation procedure. If $H(M)$ is always a quasi-firm-optimal (respectively quasi-worker-optimal) stable outcome for M, then it will be simply called a quasi-firm-optimal (respectively quasi-worker-optimal) allocation procedure. Due to the construction of the lattices $C(x)$, **a quasi-firm-optimal (respectively quasi-worker-optimal) allocation procedure always selects the firm-optimal stable payoff (respectively worker-optimal stable payoff) for the rigid and flexible markets.**

Theorem 1 and Theorem 2 together imply that workers prefer to trade in the hybrid market, rigid firms prefer the rigid market and flexible firms prefer the flexible market under any of the quasi-optimal allocation procedure. Theorem 3 is applied to the cases in which the core of at least one of the non-hybrid markets is a singleton.

Theorem 1. *Let H be a quasi-firm (respectively worker) optimal allocation procedure. Let $H(M)=(u,v;x)$ and let $H(M^F)=(u^*,v^*;x^*)$, where $M=(P,Q,a,b,c)$ and $M^F=(P^F,Q,c)$. Then, $u_i \leq u^*_i$ for every $p_i \in P^F$ and $v_j \geq v^*_j$ for every $q_j \in Q$.*

For the proof of Theorem 1 we need some technical lemmas.

Lemma 3. *Let $\sigma=(u,v;x)$ and $\sigma=(u',v';x')$ be stable outcomes for markets $M=(P,Q,a,b,c)$ and $M'=(P,Q,a',b,c)$, respectively, where $a'_{ij}<0$ for every $p_i \in P^R$ and every $q_j \neq q_0$ (i.e., only the fictitious worker is acceptable to a rigid firm in M'). Set $P^F(\sigma) \equiv \{p_i \in P^F;\ u_i > u'_i\}$ and $Q(\sigma') \equiv \{q_j \in Q;\ v'_j > v_j\}$. Then x' and x match every worker of $Q(\sigma')$ to some firm in $P^F(\sigma)$ and every firm of $P^F(\sigma)$ to some worker in $Q(\sigma')$.*

Proof. It suffices to show that $x(P^F(\sigma)) \subseteq Q(\sigma')$, $q_0 \notin Q(\sigma')$, $x'(Q(\sigma')) \subseteq P^F(\sigma)$ and then to use that x and x' are one-to-one and $P^F(\sigma)$ and $Q(\sigma')$ are finite.

To prove that $x(P^F(\sigma)) \subseteq Q(\sigma')$ take $p_i \in P^F(\sigma)$. then, $u_i > u'_i \geq 0$ and so $x(p_i)=q_j$ for some $q_j \neq q_0$. Since $p_i \in P^F$ we have that $q_j \in Q(\sigma')$, for if not $c_{ij}=u_i + v_j > u'_i + v'_j$, which contradicts the stability of $(u',v';x')$. Hence, $x(P^F(\sigma)) \subseteq Q(\sigma')$.

To see that $x'(Q(\sigma')) \subseteq P^F(\sigma)$ take $q_j \in Q(\sigma')$. Then, $v'_j > v_j \geq 0$, so $q_j \neq q_0$ and $x'(q_j)=p_i$ for some p_i, so $p_i \in P^F$ (recall every rigid firm is designated to q_0 at x'). We claim that $p_i \in P^F(\sigma)$. In fact, suppose by way of contradiction $u_i \leq u'_i$. Then,

$c_{ij}=u'_i+v'_j>u_i+v_j$, which contradicts the stability of (u,v). Hence, $x'(Q(\sigma'))\subseteq P(\sigma)$ and the proof is complete. ■

Lemma 4. *With the same hypotheses and notations of Lemma 3 we have that*

(a) $(u^+,v^-;y)$ *is stable for* $M'=(P,Q,a',b,c)$, *where* $u^+_i=\max\{u_i, u'_i\}$ *if* $p_i\in P^F$, $u^+_i=u'_i=0$ *if* $p_i\in P^R$ *and* $v^-_j=\min\{v_j,v'_j\}$ *for all* $q_j\in Q$, *and* y *agrees with* x *on* $P^F(\sigma)\cup Q(\sigma')$ *and with* x' *on the other agents;*

(b) $(u^-,v^+;y')$ *is stable for* $M=(P,Q,a,b,c)$, *where* $u^-_i=\min\{u_i, u'_i\}$ *if* $p_i\in P^F$, $u^-_i=u_i$ *if* $p_i\in P^R$ *and* $v^+_j=\max\{v_j,v'_j\}$ *for all* $q_j\in Q$, *and* y' *agrees with* x' *on* $P^F(\sigma)\cup Q(\sigma')$ *and with* x *on the other agents.*

Proof. (a) It follows the lines of the demonstration of Lemma 2-a, with the convenient adaptations, by replacing M by M' and $P(\sigma)$ by $P^F(\sigma)$. The argument only changes in the proof that (p_i,q_j) does not form a blocking pair if $p_i\in P^R$. In this case $a'_{ij}<0$ for every $q_j\in Q$, so p_i is designated to the fictitious worker at x', and therefore at y, and no other worker is acceptable to it, so (p_i,q_j) cannot be a blocking pair.

(b) It follows the lines of the proof of part (a) by reversing the roles between firms and workers. The exception is made in the proof that (p_i,q_j) does not form a blocking pair when $p_i\in P^R$. In this case, it is enough to check this fact when $v^+_j=v'_j$, since the other case is trivially satisfied by stability of $(u,v;x)$. Then, $v'_j\geq v_j$. Thus, if $v'_j<b_{ij}$ then $v_j<b_{ij}$, so p_i prefers $x(p_i)=y(p_i)$ to q_j by stability of $(u,v;x)$ and so (p_i,q_j) cannot be a blocking pair. ■

Proof of Theorem 1. Define $M'\equiv(P,Q,a', b,c)$, where $a'_{ij}<0$ for every $p_i\in P^R$ and for every $q_j\neq q_0$. Suppose first that $(u,v;x)$ and $(u^*,v^*;x^*)$ are quasi-optimal stable outcomes for firms for M and M^F, respectively**.** Let $(u',v';x')$ be a firm-optimal stable outcome for market M'. We have that $u'_i = u^*_i$ for every $p_i\in P^F$ and $v'_j=v^*_j$ for every $q_j\in Q$. According to Lemma 4-a, (u^+,v^-) is stable for M', so, using the maximality of u' and the minimality of v', it follows that $u^*_i= u'_i\geq u^+_i \geq u_i$ for every $p_i\in P^F$ and $v^*_j= v'_j\leq v^-_j\leq v_j$ for every $q_j\in Q$.

Suppose now that $(u,v;x)$ and $(u^*,v^*;x^*)$ are quasi-optimal stable outcomes for workers for M and M^F, respectively. Let $(u',v';x')$ be a worker-optimal stable outcome for M'. We have that $u'_i = u^*_i$ for every $p_i \in P^F$ and $v'_j=v^*_j$ for every $q_j \in Q$. From Lemma 4-b, $(u^-,v^+;y')$ is stable for M. The fact that $y'(p_i)=x(p_i)$ for every $p_i \in P^R$ implies that $(u^-,v^+;y') \in C(x)$. Then, using the maximality of v and the minimality of u in $C(x)$, it follows that $u_i \leq u^-_i \leq u'_i=u^*_i$ for every $p_i \in P^F$ and $v_j \geq v^+_j \geq v'_j=v^*_j$ for every $q_j \in Q$ and the proof is complete. ■

Notice that the conclusion of Theorem 1 does not change if $(u,v;x)$ is any stable outcome for M and $(u',v';x')$ is the firm-optimal stable outcome. The firm-optimal (respectively worker-optimal) stable payoff for (P,Q,a,b,c), when it exists, is quasi-optimal stable payoff for firms (respectively workers). In this case, Theorem 1 trivially implies that, **whether agents are allocated according to the firm-optimal or worker-optimal stable payoff, it will always be the case that if rigid firms enter the flexible market, no flexible firm will be made better off and no worker will be made worse off.**

Next, we address the case in which flexible firms enter the rigid market.

Theorem 2. *Let H be a quasi-firm (respectively worker) optimal allocation procedure. Let $H(M)=(u,v;x)$ and let $H(M^R)=(u^*,v^*;x^*)$, where $M=(P,Q,a,b,c)$ and $M^R=(P^R,Q,a,b)$. Then, $u_i \leq u^*_i$ for every $p_i \in P^R$ and $v_j \geq v^*_j$ for every $q_j \in Q$.*

For the proof, we will make use of Lemmas 5 and 6.

Lemma 5. *Let $\sigma=(u,v;x)$ and $\sigma=(u',v';x')$ be stable outcomes for markets $M=(P,Q,a,b,c)$ and $M'=(P,Q,a,b,c')$, respectively, where $c'_{ij}=0$ for every $p_i \in P^F$ and every $q_j \neq q_0$. Set $P^R(\sigma) \equiv \{p_i \in P^R;\ u_i > u'_i\}$ and $Q(\sigma') \equiv \{q_j \in Q;\ v'_j > v_j\}$. Then x' and x match every worker of $Q(\sigma')$ to some firm in $P^R(\sigma)$ and every firm of $P^R(\sigma)$ to some worker in $Q(\sigma')$.*

Proof. It suffices to show that $x(P^R(\sigma)) \subseteq Q(\sigma')$, $q_0 \notin Q(\sigma')$, $x'(Q(\sigma')) \subseteq P^R(\sigma)$ and then to use that x and x' are on-to-one and $P^R(\sigma)$ and $Q(\sigma')$ are finite.

To prove that $x(P^R(\sigma))\subseteq Q(\sigma')$ take $p_i\in P^R(\sigma)$. Then, $p_i\in P^R$, $u_i>u'_i\geq 0$, so p_i prefers $x(p_i)$ to $x'(p_i)$ and so p_i is designated by x to some non-fictitious worker. Let $x(p_i)=q_j$, so $v_j=b_{ij}>0$. Then, p_i prefers q_j to $x'(p_i)$. We claim that $v_j<v'_j$. In fact, if $b_{ij}=v_j=v'_j$, then $x'(q_j)\notin P^R$, since q_j has strict preferences over the rigid firms. In this case, $v'_j=0$ by definition of M', so $b_{ij}=0$, which is a contradiction. On the other hand, if $v'_j<v_j=b_{ij}$ we would have that (p_i,q_j) would block (u',v'), contradiction. Then, $v_j<v'_j$ and so $q_j\in Q(\sigma')$. Thus, all of $P^R(\sigma)$ are designated to $Q(\sigma')$ by x and the desired conclusion follows.

Now, suppose $q_j\in Q(\sigma')$. Then $v'_j>v_j\geq 0$, so q_j is not q_0 and is matched by x' to some p_i. Clearly $p_i\notin P^F$, for if so we would have $u'_i+v'_j=c_{ij}=0$, so $v'_j=0$, contradiction. We claim that $u_i>u'_i$. In fact, if $u_i=u'_i$ then rigid firm p_i is also matched to q_j under x. In this case, $v_j=v'_j$, which is a contradiction. If $u_i<u'_i$, then p_i would prefer q_j to $x(p_i)$, and so (p_i,q_j) would block (u,v), since $b_{ij}=v'_j>v_j$. But this contradicts the stability of (u,v). This way, $p_i\in P^R(\sigma)$. Hence, $x'(Q(\sigma'))\subseteq P^R(\sigma)$ and the proof is complete. ■

Lemma 6. *With the same hypothesis and notations of Lemma 5 we have that*

(a) $(u^+,v^-;y)$ *is stable for* $M'=(P,Q,a,b,c')$, *where* $u^+_i=\max\{u_i,u'_i\}$ *if* $p_i\in P^R$, $u^+_i=u'_i=0$ *if* $p_i\in P^F$ *and* $v^-_j=\min\{v_j,v'_j\}$ *for all* $q_j\in Q$, *and* y *agrees with* x *on* $P^R(\sigma)\cup Q(\sigma')$ *and with* x' *on the other agents;*

(b) $(u^-,v^+;y')$ *is stable for* $M=(P,Q,a,b,c')$, *where* $u^-_i=\min\{u_i,u'_i\}$ *if* $p_i\in P^R$, $u^-_i=u_i$ *if* $p_i\in P^F$ *and* $v^+_j=\max\{v_j,v'_j\}$ *for all* $q_j\in Q$, *and* y' *agrees with* x' *on* $P^R(\sigma)\cup Q(\sigma')$ *and with* x *on the other agents.*

Proof. (a) To see that $(u^+,v^-;y)$ is feasible, first observe that Lemma 5 implies that y is a feasible matching. Conditions a), b) and c) of Definition 1 follow from the feasibility of $(u,v;x)$ and $(u',v';x')$. Therefore, it remains to show that if $y(p_i)=q_j$ then d) $u^+_i+v^-_j=c_{ij}$ if $p_i\in P^F$ and e) $u^+_i=a_{ij}$, $v^-_j=b_{ij}$ if $p_i\in P^R$. Part d) is immediate since $q_j=y(p_i)=x'(p_i)$, $u^+_i=u'_i=0$, $c_{ij}=0$, so $v'_j=0=v^-_j$. For part e), if $p_i\in P^R(\sigma)$ then $q_j\in Q(\sigma')$, so $y(p_i)=x(p_i)$, $u^+_i=u_i$ and $v^-_j=v_j$, so $u^+_i=a_{ij}$ and $v^-_j=b_{ij}$. If $p_i\notin P^R(\sigma)$ then $q_j\notin Q(\sigma')$, so $y(p_i)=x'(p_i)$, $u^+_i=u'_i$ and $v^-_j=v'_j$, so $u^+_i=a_{ij}$ and $v^-_j=b_{ij}$.

That $(p_i,q_j)\in PxQ$ does not form a blocking pair is immediate if $p_i\in P^R$ and $u^+_i=u_i$ and $v^-_j=v_j$ or $u^+_i=u'_i$ and $v^-_j=v'_j$ or if $p_i\in P^F$. In this last case, for all $q_j\in Q$, we have that $c'_{ij}=0$, so $u^+_i + v^-_j \geq 0=c'_{ij}$ and so (p_i,q_j) cannot form a blocking pair. For the other cases, if $p_i\in P^R$, $u^+_i=u_i$ and $v^-_j=v'_j$ then $u_i \geq u'_i$. If $v'_j <b_{ij}$, the stability of $(u',v';x')$ implies that $u'_i\geq a_{ij}$, so $u_i\geq a_{ij}$ and so $u^+_i\geq a_{ij}$. If $p_i\in P^R$, $u^+_i=u'_i$ and $v^-_j=v_j$ then $u'_i \geq u_i$. If $v_j<b_{ij}$, use the stability of $(u,v;x)$ to see that $u_i\geq a_{ij}$, so $u'_i\geq a_{ij}$ and so $u^+_i\geq a_{ij}$. Therefore, in any case, we do not have a blocking pair. Hence, $(u^+,v^-;y)$ is stable for M'.

(b) It follows the lines of the proof of part (a), by reversing the roles between firms and workers. The exception is made in the proof that (p_i,q_j) does not form a blocking pair when $p_i\in P^F$. In this case, $u^-_i + v^+_j =u_i+v^+_j\geq u_i+v_j\geq c_{ij}$ and so (p_i,q_j) cannot form a blocking pair. Therefore, in any case, we do not have a blocking pair. Hence, $(u^-,v^+;y')$ is stable for M. ■

Proof of Theorem 2. Define $M'\equiv(P,Q,a,b,c')$, where $c'_{ij}=0$ for every $p_i\in P^F$ and for every $q_j\neq q_0$. Let $\sigma'=(u',v';x')$ be a firm-optimal stable payoff for market M'. Set $\sigma\equiv(u,v;x)$. Obviously $u'_i = u^*_i$ for every $p_i\in P^R$ and $v'_j=v^*_j$ for every $q_j\in Q$. According to Lemma 6-a, (u^+,v^-) is stable for M', so, using the maximality of u' and the minimality of v', it follows that $u^*_i= u'_i\geq u^+_i \geq u_i$ for every $p_i\in P^R$ and $v^*_j= v'_j\leq v^-_j\leq v_j$ for every $q_j\in Q$.

For the other assertion, suppose *(u',v';x')* is a worker-optimal stable outcome for M'. Define $P^R(\sigma)$ and $Q(\sigma')$ as in Lemma 5. From Lemma 6-b, $(u^-,v^+;y')$ is stable for M and $y'(P^R(\sigma))=x'(P^R(\sigma))$. From Lemma 5, $x(P^R(\sigma))=x'(P^R(\sigma))=Q(\sigma')$. Then, $y'(P^R(\sigma))=x(P^R(\sigma))$. On the other hand, $y'(P^R-P^R(\sigma))=x(P^R-P^R(\sigma))$. Then, $(u^-,v^+;y')\in C(x)$. Then, using the maximality of v and the minimality of u in C(x), it follows that $u_i\leq u^-_i \leq u'_i=u^*_i$ for every $p_i\in P^R$ and $v_j\geq v^+_j\geq v'_j =v^*_j$ for every $q_j\in Q$, and the proof is complete. ■

A more general result can be obtained when the core of one of the non-hybrid markets is a singleton:

Theorem 3. *Let H be a stable allocation procedure. Let $H(M)=(u,v;x)$, let $H(M^R)=(u^1,v^1;x^1)$ and let $H(M^F)=(u^2,v^2;x^2)$, where $M=(P,Q,a,b,c)$, $M^R=(P^R,Q,a,b)$ and $M^F=(P^F,Q,a,b)$. Then, (a) if the core of the rigid market is a singleton, $u_i\leq u^1_i$ for every $p_i\in P^R$ and $v_j\geq v^1_j$ for every $q_j\in Q$; (b) if the core of the flexible market is a singleton then, $u_i\leq u^2_i$ for every $p_i\in P^F$ and $v_j\geq v^2_j$ for every $q_j\in Q$.*

Proof. We are going to prove statement (a). The proof of statement (b) is analogous and is left to the reader. Define $M'\equiv(P,Q,a,b,c')$, where $c'_{ij}=0$ for every $p_i\in P^F$ and every $q_j\neq q_0$. Let (u',v') be the only stable payoff for market M'. It is clear that $u^1_i=u'_i$ for every $p_i\in P^R$ and $v^1_j=v'_j$ for every $q_j\in Q$. Define $(u^+,v^-;y)$ as in Lemma 6-a. This result implies that $(u^+,v^-;y)$ is stable for M'. Since the core of the rigid market is a singleton, it follows that $(u',v';x')$ is the only stable outcome of M'. Then, $(u',v';x')=(u^+,v^-;y)$ and so $u'_i=u^+_i\geq u_i$ and $v'_j=v^-_j\leq v_j$ for all $(p_i,q_j)\in P^R xQ$. Then, $u_i\leq u^1_i$ for every $p_i\in P^R$ and $v_j\geq v^1_j$ for every $q_j\in Q$. ■

5. FURTHER RESULTS.

The algebraic structure of the core of the hybrid market is not useful to approach the general problem of obtaining meaningful comparative static results of adding agents to that market. In fact, it is easy to construct examples where the comparison between two quasi-optimal stable outcomes for the same side of the hybrid market may be meaningless. By using the framework of simple outcomes from Sotomayor (2000) given below, we are able to prove a partial result that can be a starting point for future investigations and it is implied by Theorem 4: *When any finite subset of firms (respectively, workers) is added to the pool of the hybrid market, and the players are allocated according to some quasi-optimal stable outcome for workers (respectively, firms), then there is some quasi-optimal stable outcome for workers (respectively, firms) in the core of the new market under which no worker (respectively, firm) is worse off and no flexible firm (respectively, worker matched to a flexible firm) is better off. If the stable outcome does not have any weak blocking pair then no rigid firm (respectively, worker) is better off.*

Definition 5. *The outcome $(u,v;x)$ for (P,Q,a,b,c) is a* ***P-simple outcome*** *if it is feasible and every blocking pair, if any, is formed with an unmatched firm. Analogously we define a* ***Q-simple outcome.***

Thus, if $(u,v;x)$ is P-simple then no matched firm forms a blocking pair. The set of P (respectively, Q)-simple outcomes is non-empty, since the outcome where every player is unmatched is P (respectively, Q)-simple.

Definition 6. *Consider the market for (P,Q,a,b,c). We say that the P-simple outcome $(u',v';x')$* ***extends the P-simple outcome $(u,v;x)$*** *if (a) $v'_j \geq v_j$ for all $q_j \in Q$, with strict inequality for at least one player in Q or (b) $v'=v$ and $(u',v';x')$ has less blocking pairs than $(u,v;x)$.*

The payoff $(u,v;x)$ is a ***Pareto optimal P-simple outcome for workers*** *if it is P-simple and if it does not have any P-simple extension. Analogously we define a* ***Pareto optimal Q-simple outcome for firms.***

Set **$S_Q \equiv \{v \in R^n$; $(u,v;x)$ is a P-simple outcome, for some $u \in R^m$ and matching $x\}$.** It follows from the construction of the P-simple outcomes that S_Q is a compact set of R^n. Then there is some Pareto optimal P-simple outcome for workers. Analogously, there is some Pareto optimal Q-simple outcome for firms. The proof of Theorem 3 makes use of the following result implied by Theorem 1 of Sotomayor (2000):

Proposition 1. *Any Pareto optimal P-simple outcome for workers, as well as any Pareto optimal Q-simple outcome for firms, is in the core.*

Theorem 4. *Let $(u,v;x)$ be some stable outcome for the hybrid market $M=(P,Q,a,b,c)$. (a) Suppose $p_k \in P$ is added to M. Then, there exists some quasi-optimal stable outcome for workers, $(u^*,v^*;x^*)$, for the new market M^*, such that $v^*_j \geq v_j$ for every $q_j \in Q$ and $u^*_i \leq u_i$ for every $p_i \in P^F$. If no $p_i \in P^R$ is part of a weak blocking pair of $(u,v;x)$ then $u^*_i \leq u_i$ for every $p_i \in P$.*

(b) Suppose $q_k \in Q$ *is added to* M*. Then, there exists some quasi-optimal stable outcome for firms,* $(u^{**},v^{**};x^{**})$ *for the new market* M^{**}*, such that* $u^{**}_i \geq u_i$ *for every* $p_i \in P$ *and* $v^{**}_j \leq v_j$ *for every* $q_j \in x(P^F)$*. If no* $p_i \in P^R$ *is part of a weak blocking pair of* $(u,v;x)$ *then* $v^{**}_j \leq v_j$ *for every* $q_j \in Q$*.*

Proof. We are going to prove (a). Statement (b) can be proved with analogous arguments, by exchanging the roles between P and Q in the proof. Then, let $(u',v';x')$ be the outcome that agrees with $(u,v;x)$ on $P \cup Q$ and leaves p_k unmatched with payoff $u'_k=0$. This outcome is clearly feasible for M^*. On the other hand, if $(u',v';x')$ is unstable then every blocking pair must contain p_k. Then, $(u',v';x')$ is a P-simple outcome. Set $A \equiv \{v'' \in S_Q;\ v''_j \geq v'_j \text{ for all } q_j \in Q\}$. We have that $A \neq \phi$, since $v' \in A$. Clearly, A is a compact subset of R^n, so there is some P-simple outcome $(u^*,v^*;x^*)$ such that $v^* \in A$ and

$$\sum_{j \in Q} v^*_j \geq \sum_{j \in Q} v''_j \text{ for all } v'' \in A. \qquad (1)$$

Without loss of generality we can choose u^* and x^* so that $(u^*,v^*;x^*)$ has the minimum number of blocking pairs among all P-simple outcomes that are compatible with v^*. We are going to show that $(u^*,v^*;x^*)$ is a Pareto optimal P-simple outcome for workers. If this is established, Proposition 1 then implies that $(u^*,v^*;x^*)$ is in the core. In addition, we will have that $v^*_j \geq v_j$ for all $q_j \in Q$. Then, if $(u^*,v^*;x^*)$ was not Pareto optimal P-simple outcome for workers, then there would be some P-simple outcome $(u'',v'';x'')$ that would extend $(u^*,v^*;x^*)$. Then we should have that $v''_j \geq v^*_j \geq v'_j$ for all $q_j \in Q$, so $v'' \in A$. On the other hand, since $(u^*,v^*;x^*)$ has the minimum number of blocking pairs among all P-simple outcomes that are compatible with v^*, it follows that $v'' > v^*$. But this contradicts (1). Hence, $(u^*,v^*;x^*)$ is in the core and $v^*_j \geq v_j$ for all $q_j \in Q$. Now use that $(u^*,v^*;x^*) \in C(x^*)$ and $C(x^*)$ is a lattice, to see that $(u^*,v^*;x^*)$ is the quasi-optimal stable outcome for workers of $C(x^*)$.

Now, let $p_i \in P$. Suppose $u^*_i > u_i$. Then there exists some $q_j \in Q$ such that $x^*(p_i)=q_j$. Using that $v^*_j \geq v_j$ it follows that $c_{ij}=u^*_i+v^*_j > u_i+v_j$, so (p_i,q_j) blocks $(u,v;x)$ in case $p_i \in P^F$, which is a contradiction. Hence $u_i \geq u^*_i$ for all $p_i \in P^F$.

In case $p_i \in P^R$ we cannot have $v^*_j > v_j$, for if so $(u,v;x)$ is blocked by (p_i,q_j), contradiction. Then, $v^*_j=v_j$, so (p_i,q_j) weakly blocks $(u,v;x)$, contradiction.

Hence $u_i \geq u^*_i$ for all $p_i \in P^R$ and the proof is complete. ■

6. FINAL REMARKS, NEW DIRECTIONS AND RELATED WORK

Questions of comparative static naturally emerge when we face a two-sided matching market. Without seeking wide generality, this paper has concentrated in a quite specific problem of economic interest*: If rigid firms enter the flexible market or flexible firms enter the rigid market, is that good or bad and to whom?*

Of course, the comparison between arbitrary core points of these markets may be meaningless. In the literature, meaningful comparative static results of adding agents to matching markets have always been obtained under the assumption that agents are allocated according to the firm-optimal or worker-optimal stable payoff. However, we observed that the main properties that characterize the core of a non-hybrid market do not, in general, apply to the hybrid model. The core of the hybrid market is not always a complete lattice and the firm-optimal and worker-optimal stable payoffs not always exist.

We solved this problem by identifying a non-standard algebraic structure of the core of the hybrid market, described by the disjoint union of quite specific complete lattices. An important feature of such characterization is that the lattice structure and the polarization of interests found in the cores of the non-hybrid markets persist, in a consistent way, for these subsets of the core of the hybrid market. More specifically, as all flexible (respectively, rigid) firms leave the hybrid market, such algebraic structure coincides with the complete lattice structure of the core of the Marriage model (respectively, Assignment model). Then, all extreme points of the lattices of the core partition, called here quasi-firm-optimal and quasi-worker-optimal stable payoffs, coincide with the respective extreme points of the lattices of the non-hybrid models. A consequence of this property is that the distinctions between any quasi-firm-optimal (respectively, quasi-worker-optimal) stable payoff and the firm-optimal (respectively worker-optimal) stable payoff for, say, the marriage model, are consistent with the entrance of flexible firms in the rigid market or, equivalently, with the exit of flexible firms from the hybrid market. Without this property, the comparative static may be meaningless, as showed by Example 4. As a result of our investigation, we demonstrated that workers weakly prefer to negotiate with both types of firms in the hybrid market while rigid firms weakly prefer the rigid market and flexible firms weakly prefer the flexible market, whatever particular quasi-optimal stable allocation procedure is being used.

To our knowledge, our results are the strongest ones in terms of comparative statics in one-to-one matching markets. Due to the intrinsic characteristics of the algebraic structure of the core of the hybrid market, a new kind of comparative static, that has no parallel in the discrete and continuous models, could then be observed:

1.*The firms that are added are different from the firms which are already in the market.* For example, in the marriage market, where utility is non-transferable, the comparative static adds firms with flexible wages who can transfer utility.

2. *Our results compare points from cores with quite distinct algebraic structures.* The points we pick in the core of the hybrid model may be neither the optimal stable payoff for firms nor the optimal stable payoff for workers, even when these outcomes exist. But those ones we pick in the cores of the non-hybrid markets are always optimal for one of the sides.

3. *We may have several quasi-optimal stable outcomes for firms and several quasi-optimal stable outcomes for workers. In despite of the multiplicity of these outcomes, all of them reveal the same kind of comparative static effects*. More specifically, no firm is better off and no worker is worse off under any of the quasi-optimal stable outcomes for firms (respectively, workers) than under the firm-optimal (respectively, worker-optimal) stable outcome for the Marriage market or for the Assignment market.

The main feature of the core partition is that, as all flexible (respectively, rigid) firms leave the hybrid market, the number of lattices in the partition decreases to one. An interesting mathematical problem is to know if the number of lattices in the core partition decreases monotonically to one, as the number of flexible (respectively, rigid) firms decreases. It is known that the multiplicity of sets in the core partition is due to the existence of weak blocking pairs. When no weak blocking pairs exist, the core is a complete lattice (Sotomayor, 2000). Then, it seems that the proposed problem is related to the existence of some kind of weak blocking pair.

Another problem of mathematical interest is to find necessary conditions on the preferences of the agents under which the core of the hybrid market is a complete lattice. A sufficient condition, for example, is obtained by requiring that any rigid firm offers a salary that is bigger than any salary that a flexible firm can pay.

Results of comparative statics were originally obtained by Gale and Sotomayor (1985-b) for the Marriage market and by Demange and Gale (1985) for a continuous one-to-one matching model that includes the Assignment game, where utilities are not necessarily linear. For the Assignment Game, Shapley (1962) showed that the optimal stable payoff for an agent weakly decreases when another agent is added to the same side and weakly increases when another agent is added to the other side. Still with regard to the Assignment Game, Mo (1988) showed that if the incoming worker is allocated to some firm in some stable outcome for the new market, there is a set of agents such that every firm is better off and every worker is worse off in the new market than in the previous one. Symmetric result is valid when the incoming agent is a firm. An analogous result is demonstrated by Roth and Sotomayor (1990) for the Marriage Market.

For the many-to-one matching markets, Kelso and Crawford (1982) showed that, within the context of (flexible) firms and workers, the addition of one or more firms to the market weakly improves the workers' payoffs, and the addition of one or more workers weakly improves the firms' payoffs, under the firm-optimal stable allocation. Similar conclusions were obtained by Crawford (1991) for a many-to-many matching model, by comparing pairwise-stable outcomes instead of stable outcomes.

REFERENCES

Birkhoff G., (1973), “Lattice Theory”, third ed. *American Mathematical Society Coloquium Publications,* vol. 25, *American Mathematical Society,* Providence.

Bohm-Bawerk, E.Von (1923, originally published in1891), "Positive theory of capital", G.E. Steckert. New York.

Clark S. (2003), "Uniqueness of Equilibrium in Two-Sided Matching, mimeo, University of Edinburgh.

Cournot, A. A. (1897, originally published in 1838,) "Researches into the mathematical principles of the theory of wealth", Macmillan Co., New York.

Crawford, V.P. (1991), "Comparative statics in matching markets", *Journal of Economic Theory*, 54, 389-400.

Demange, G., (1982), "Strategyproofness in the assignment market game". Laboratoire d'Econometrie de l'École Polytechnique, Paris, Mimeo.

Demange, G. and Gale. D; (1985), "The strategy structure of two-sided matching markets", *Econometrica*, 53, 873-88.

Demange, G., Gale, D. and Sotomayor, M. (1986), "Multi-item auctions", *Journal of Political Economy,* 94, 863-72.

Edgeworth, F. Y. (1881), " Mathematical psychics", Keagan Paul.

Eeckout, J. (2000), "On the uniqueness of stable marriage matchings", *Economics Letters* 69, 1-8.

Ehlers, L. and Masso, J. (2004), "Incomplete information and small cores in matchings markets", *mimeo.*

Eriksson, K. and Karlander, J. (2000) "Stable matching in a common generalization of the marriage and assignment models", *Discrete Mathematics,* 217, 135-156.

Gale, D. and Shapley, L. (1962) "College admissions and the stability of marriage", *American Mathematical Monthly*, 69, 9-15.

Gale, D. and Sotomayor, M. (1985-a), "Some remarks on the stable matching problem", *Discrete Applied Mathematics* 11, 223-232.

______________________ (1985-b) "Ms. Machiavelli and the stable matching problem", *American Mathematical Monthly*, 92, 261-8.

Kelso, A. S. Jr. and Crawford V.P. (1982) "Job matching, coalition formation, and gross substitutes", *Econometrica,* 50, 1483-1504.

Knuth, D. E. (1976), "Marriages Stables", Montreal: Les Presses de l'Université de Montreal.

Leonard, H.,(1983) " Elicitation of honest preferences for the assignment of individuals to positions", *Journal of Political Economy,* 91, 461-79.

Mo, J.P., (1988) " Entry and structures of interest groups in assignment games", *Journal of Economic Theory",* 46, 66-96.

Roth, A. and Sotomayor, M. (1990) "Two sided matching. A study in game theoretic modeling and analysis", *Econometric Society Monographs*, 2nd ed., Vol. 18, Cambridge University Press.

___________________________(1996) "Stable outcomes in discrete and continuous models of two-sided matching: a unified treatment", *The Brazilian Review of*

Econometrics, v. 16, n^0 2, 1-24.

Shapley, L. S., (1962) "Complements and substitutes in the optimal assignment problem", *Naval Research Logistics Quarterly,* 9, 45-8.

Shapley, L. S. and Shubik, M. (1972) "The assignment game I: The core". *Rigid Journal of Game Theory* 1, 111-130.

Sonmez, T. (1999) "strategy-proofness and essentially single-valued cores", *Econometrica* 67, 677-689.

Sotomayor M.(1996) "A non-constructive and elementary proof of the existence of stable marriages", *Games and Economic Behavior,* 13,135-137.

________________(2000) "Existence of stable outcomes and the lattice property for a unified matching market". *Mathematical Social Sciences*, 39, 199-132.