ADJUSTING PRICES IN THE MANY-TO-MANY ASSIGNMENT GAME TO YIELD THE SMALLER COMPETITIVE EQUILIBRIUM PRICE VECTOR

By

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ABSTRACT

Shapley and Shubik (1972) incorporated money into the Marriage model of Gale and Shapley (1962) to establish the well-known Assignment game. Since then, this model has been widely studied and generalized to more complex models. In Sotomayor (2006), an economic structure in terms of buyers and sellers was proposed for a many-to-many extension. The set of competitive equilibrium payoffs was shown to be a proper sub-lattice of the set of stable payoffs, and the connection between the corresponding extreme points of these lattices was established. The present paper provides a dynamic procedure for finding these extreme points for both lattices. The manipulability questions that emerge are discussed.

Key words: matching, stable payoff, competitive equilibrium payoff, optimal stable payoff, lattice

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INTRODUCTION

The many-to-many Assignment game of our title is one of the two matching models with additively separable utilities introduced in Sotomayor (1992). In this game there are two finite and disjoint sets of players, B and Q. A player of a set may form more than one partnership with different players of the other set. Every participant has a quota representing the maximum number of partners. The main characterization of this game is that players negotiate their individual payoffs: If b and q become partners, they undertake an activity together that produces a gain v_{bq}, which is divided between them the way both agree: the individual payoff $u_{bq}\geq 0$ for b and the individual payoff $w_{bq}=v_{bq}-u_{bq}\geq 0$ for q. Therefore, an outcome is a set of partnerships that does not violate the quotas of the players, plus an array of individual payoffs for each player.[2] The Assignment Game, formulated and studied by Shapley and Shubik (1972), is the well-known special case, where the only restriction is that each agent is allowed to form one partnership at most.

A way of interpreting this game is to think of it as being a labor market of firms and workers operating cooperatively. The quota of a firm is the maximum number of workers it can hire; the quota of a worker is the maximum number of jobs he/she can take. The number v_{bq} is the productivity of worker q in firm b. The appropriate cooperative equilibrium concept is the one of setwise-stability (stability for short)[3], defined in the text. Unlike the one-to-one case, setwise-stable outcomes are not envy-free, since an agent may receive different payments from different partners.

Another interpretation is obtained by approaching the game competitively through a market of buyers and sellers: B is a set of buyers and Q is a set of sellers. Buyers are interested in sets of objects owned by different sellers and each seller owns a number of

[2] Another approach is treated in Sotomayor (1992). Agents are not allowed to negotiate their individual payments and act in blocks. An outcome only specifies their total payoffs. The cooperative equilibrium concept is given by the core concept.

[3] Setwise-stability captures the intuitive idea that instabilities may be caused by coalitions of any size. Since agents can negotiate their individual payments, in order to check instabilities we only need to consider pairs of agents made up with a firm and a worker. Thus, in this model, setwise-stability is equivalent to pairwise-

identical objects. The quota of a buyer is the number of objects she is allowed to acquire; the quota of a seller is the number of identical objects he owns. The number v_{bq} is the maximum amount of money buyer b considers paying for an object of seller q. Given the prices of the objects, buyers demand their favorite sets of objects. The adequate solution concept is that of competitive equilibrium payoff, introduced in Sotomayor (2006). Roughly speaking, the payoff (u,w) is a *competitive equilibrium payoff* if there is a feasible allocation, μ, under which each active buyer receives one of her demanded sets of items (i.e. a set of items that, given prices, maximizes her additive utility payoff), every inactive buyer has a zero payoff and every unsold object has zero price. The pair (w, μ) is called a competitive equilibrium and w a competitive equilibrium price vector. (The formal definition is given in the text). Unlike the setwise-stable payoffs, the competitive equilibrium payoffs are envy-free on the part of buyers: every seller sells all his objects at the same competitive equilibrium price, so no buyer envies another buyer.

Sotomayor (2006) shows that the set of competitive equilibrium payoffs is non-empty and is endowed with a complete lattice structure under convenient partial order relations. Although these partial orders are not defined by the preferences of the players, the extreme points of these lattices have important meaning for the market. They reflect a surprising coincidence of interest among agents on the same side of the market, and a corresponding conflict of interest among agents on opposite sides. All buyers, as well as all sellers, agree on the best competitive equilibrium payoff for them. These outcomes are called *B*-optimal competitive equilibrium payoff and *Q*-optimal competitive equilibrium payoff, respectively. In addition, the *B*-optimal competitive equilibrium payoff (respectively, *Q*-optimal competitive equilibrium payoff) is the worst competitive equilibrium payoff under the point of view of the sellers (respectively, buyers).[4] That is, the corresponding payoff vector of the sellers is the minimum (respectively, maximum) competitive equilibrium price vector in the sense that it is smaller (respectively, greater) in each component than any other competitive equilibrium price vector. The existence of these outcomes is of great applicability in the organization of real markets, because of the

stability.

[4] Shapley and Shubik [21] prove these results for the one-to-one Assignment Game. See also Roth and Sotomayor [20] for an overview of this model.

close relation between the optimal stable mechanism for one side and its non-manipulability by agents of that side with a quota of one.

The results above are similar to a characterization of the set of stable payoffs, as obtained in Sotomayor (1999-a). Unlike the one-to-one case, the lattice of the competitive equilibrium payoffs is a proper sub lattice of the lattice of the stable payoffs. The *B*-optimal stable payoff equals the *B*-optimal competitive equilibrium payoff, but the *Q*-. optimal competitive equilibrium payoff may be different from the *Q*-optimal stable payoff. The *Q*-optimal competitive equilibrium payoff can be obtained from the *Q*-optimal stable payoff by reducing the price of each of the items of every seller to his minimal individual payoff.

The present paper provides a simple procedure to finding the optimal stable and the optimal competitive payoffs, for each side of the market. The *B*-optimal stable payoff and consequently the *B*-optimal competitive equilibrium payoff are achieved by a dynamic mechanism. When every agent has a quota of one, this procedure coincides with the dynamic auction of Demange, Gale and Sotomayor (1986)[5]. The key notion in both auctions is that of an over-demanded set. For the quota one case, a set of items is over-demanded if the number of buyers demanding only objects in the set is greater than the number of objects in the set. It is minimal if it does not have any proper over-demanded subset. The general definition of an over-demanded set is given in the text.

Our dynamic mechanism can be roughly described as follows: At the first step, given the reservation prices to the sellers announced by the auctioneer, each buyer indicates his/her demand according to those prices. If it is possible to satisfy the demand of every buyer, respecting sellers' quotas, then the auction stops. Otherwise, the auctioneer selects a convenient *minimal over-demanded set* of objects (there may be more than one such a set). The auctioneer, then, raises the prices of every item in this set by one unit. All other prices remain the same as those in the previous step, and so on.[6] This algorithm stops after a finite number of steps, because as soon as the price of an object becomes higher than any buyer's

[5] This mechanism is a version of the Hungarian algorithm (see Dantizig, 1963) and a generalization of the English auction for a single object.

[6] The idea of raising the prices of every object in some convenient over-demanded set has been also explored by Gul and Stacchetti (2000). These authors get a more complex definition of over-demanded set, for an economy where all objects are distinct and are owned by the same seller. Utilities satisfy the monotonicity property and the gross substitute assumption of Kelso and Crawford (1982). The mechanisms of Demange *et al.* and of Gul *et al.* also yield the minimum competitive equilibrium payoff.

valuation for it, no buyer demands it. The final outcome is competitive by the construction of the mechanism. The conclusion that this outcome is a competitive equilibrium outcome and does not depend on the particular choices of the auctioneer along the algorithm is due to the fact proved here that the mechanism always yields the *B*-optimal competitive equilibrium payoff. The intuition for this result is that the mechanism increases the prices of the items as long as there is excess demand, therefore producing a competitive equilibrium. At the same time, it keeps the transfers to the sellers as low as possible, thus yielding the smaller competitive equilibrium price. By reversing the roles between buyers and sellers, a dual mechanism produces the *Q*-optimal stable payoff. The *Q*-optimal competitive equilibrium payoff can then be derived as described above.

Manipulability questions are naturally raised when we adopt an allocation mechanism. The allocation mechanism that produces the optimal stable outcome for one side is manipulable by the agents of the opposite side (Demange and Gale, 1985). For the direct mechanism implemented by Demange *et al.*(1986), Demange (1982) and Leonard (1981) showed that truthful behavior constitutes a dominant strategy for each buyer. Our main finding is that the many-to-one Assignment game (buyers have a quota of one) is the most general Assignment game[7] for which a dynamic mechanism that yields the buyer-optimal stable outcome is strategy-proof for the buyers. In fact, an example in the present paper shows that this mechanism is manipulable by the buyers when multiplicity of partnerships in both sides of the market is allowed. On the other hand, when every buyer is interested in acquiring one object at most, this mechanism is strategy-proof for the buyers. More generally, if every buyer has a quota of one, no coalition of buyers can, by misrepresenting their demands, get a *true* outcome which all of its members prefer to the *true* optimal stable payoff for the buyers. An additional result proves that, when every buyer wants to acquire all objects in the market, this mechanism makes it a dominant strategy for each buyer to honest reveal his/her demands.

The present article is structured as follows. Section 2 gives the cooperative framework. Section 3 is devoted to the dynamic mechanism. The concepts and terminology needed for the description of the mechanism and for the proofs of the main results are introduced. Section 3.1 describes the mechanism. An illustrative example is given in

[7] By an Assignment game we mean one of the three games: the one-to-one Assignment game of Shapley and

section 3.2. The main results are in Section 4. Section 5 discusses the manipulability results. Section 6 gives some final remarks and discusses related works. Some of the proofs are presented in the Appendix.

2.THE FRAMEWORK

The many-to-many matching model discussed in the Introduction consists of two finite and disjoint sets of players, B and Q, which we can think of as being buyers and sellers, respectively. The B-players may form more than one partnership with Q-players, and Q-players may form more than one partnership with B-players. The set B has m elements and the set Q has n elements. Each $q \in Q$ has a quota $s(q)$ and each $b \in B$ has a quota $r(b)$, representing the maximum number of partnerships they can form. Quota $s(q)$ of seller q means that q owns $s(q)$ identical and indivisible objects, and quota $r(b)$ of buyer b represents the maximum number of objects buyer b is allowed to buy. Without loss of generality we can consider $r(b) \leq n$ and $s(q) \leq$m. No buyer is interested in acquiring more than one item of a given seller.

Generically, we will denote buyers by b, b', and sellers and objects by q, q'. Every object has a reservation price of 0 *(which can be obtained after normalization)*. For each pair (b,q) there is a non-negative number $v_{bq} \geq 0$ which is split between b and q if both form a partnership. We will be assuming that agents' preferences over potential partners are *separable across pairs*, in the sense that the payoff of the partnership (b,q), v_{bq}, does not depend on which other partnerships are formed by buyer b and seller q. We can interpret this number as the value of any object of seller q to buyer b. That is, v_{bq} is the gain of trade when any of the objects of seller q is sold to buyer b. If buyer b acquires some object of q at price π then b receives the individual payoff $u_{bq} \equiv v_{bq} - \pi$.

Dummy players, denoted by 0, are included for technical convenience in both sides of the market. We have that $v_{b0} = v_{0q} = 0$ for all $b \in B$ and $q \in Q$. As for the quotas, a dummy player may form as many partnerships as needed to fill up the quotas of the non-dummy players. We will also include an artificial "null-object", 0, owned by the artificial seller, whose value is zero to all buyers and whose price is always zero.

We will say that a subset $S \subseteq Q$ is an **allowable set of partners for** $\boldsymbol{b \in B}$, if $|S|$

Shubik , the many-to-one Assignment game or the many-to-many Assignment game.

$=r(b)$. We will extend this terminology to include the sets S with k non-dummy players and $r(b)$-k repetitions of the dummy player for $0 \leq k \leq r(b)$. Analogously, we define an **allowable set of partners for $q \in Q$.** In order to simplify notation, we will also write $S \subseteq B$ or $S \subseteq Q$ for any allowable set S of B-players or Q-players, respectively. **An allowable set of objects for buyer b contains $r(b)$ objects**, some of which may be repetitions of the null-object. Furthermore, it does not contain more than one object of the same seller (an exception is made to the fictitious seller).

Under the cooperative approach, agents b and q may form a partnership and negotiate their individual payments u_{bq} and w_{bq} the way they like. A ***matching*** μ is a set of partnerships of the kind (b,q), $(b,0)$ or $(0,q)$, for $(b,q) \in B \times Q$. If b and q are matched under μ, we write $b \in \mu(q)$ or $q \in \mu(b)$. A dummy player may be matched to more than one player of the opposite side and more than once to the same player.

We define **a matching μ to be feasible if each player is matched to an allowable set of partners.** The value of μ is $\sum_{q \in Q, b \in \mu(q)} v_{bq}$. The matching μ is **optimal** if it attains the maximum value among all feasible matchings.

For any sets $R \subseteq B$ and $S \subseteq Q$,, the payoff $P(R \cup S)$ of coalition $R \cup S$ is the maximum $\sum_{b \in R, \mu(b) \in S} v_{b\mu(b)}$ over all feasible matchings μ.

If S is an allowable set of partners for $b \in B$, **the payoff $P(\{b\} \cup S)$ of the coalition $\{b\} \cup S$** is the sum of the numbers v_{bq}'s with $q \in S$. Similarly, we define **the payoff $P(\{q\} \cup S)$,** where S is an allowable set of partners for $q \in Q$. In this paper we will refer to functions $P(\{b\} \cup \{.\})$ and $P(\{q\} \cup \{.\})$ as being *additively separable*.

This cooperative game will be denoted by M and is more appropriate to model a job market of firms and workers.[8] In this case the numbers v_{bq}'s can be interpreted as being the productivity of worker q in firm b, which is divided between the agents into salary w_{bq} and net profit u_{bq}.

For our purposes we define:

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[8] This model was introduced in Sotomayor (1992) and has also been treated in Sotomayor (1999). It is a version of the model of Crawford and Knower (1981) and is the simplest extension of the Assignment game of Shapley and Shubik (1972) to the case of multiple partners.

Definition 1. *A* ***feasible outcome for M,*** *denoted by* $(u,w;\mu)$*, is a feasible matching* μ *and a pair of payoffs* (u,w)*, where the individual payoffs of each* $b\in B$ *and* $q\in Q$ *are given by the arrays of numbers* u_{bq}*, with* $q\in\mu(b)$*, and* w_{bq}*, with* $b\in\mu(q)$*, respectively, such that* $u_{bq} + w_{bq}=v_{bq}$*,* $u_{bq}\geq 0$ *and* $w_{bq}\geq 0$*. Consequently,* $u_{b0}=u_{0q}=w_{b0}=w_{0q}=0$ *in case these payoffs are defined.*

If $(u,w;\mu)$ is a feasible outcome, we say that μ is compatible with payoff (u,w) and vice-versa.

Under the competitive approach, a **feasible outcome** is a feasible allocation of the objects to the buyers plus a non-negative price for each object and a corresponding array of non-negative individual payoffs for each buyer. **A feasible allocation allocates each non-null object to one buyer** (who might be the dummy buyer) so that **each non-dummy buyer is assigned to an allowable set of objects for her.** If an object is allocated to the dummy buyer we say that it is left unsold. Of course, a dummy buyer may be assigned to any number of objects and the null object may be allocated to any number of buyers.

If an object is allocated to a buyer then the seller who owns this object is matched to that buyer. Thus, if μ^* is a feasible allocation, we can define a corresponding matching μ such that seller $q\in\mu(b)$ if and only if one of his objects is allocated to b under μ^*[9]. We say that μ and μ^* **correspond** to each other. Clearly, μ and μ^* have the same value, so μ **is an optimal matching if and only if** μ^* **is an optimal allocation.**

A vector of prices $p\in R^N_+$, with $N\equiv\sum_{q\in Q} s(q)$, is called a **feasible price vector** or **price vector**, for short**.** Given a price vector, the preference of a buyer over sets of objects is defined only over allowable sets of objects. Due to the characteristics of the market, there is no loss by assuming that. **The value of an allowable set of objects** ***S*** **to buyer** ***b*** is the sum of the values of the objects in S to b. That is, it is the payoff of the coalition consisting of b and the sellers who own the objects in S. Then, given a price vector, the preferences of buyers over objects are completely described by the numbers v_{bq}*'s:* For any

[9] If μ^* is a feasible allocation and a buyer buys an object from a seller, then make a link between the two. The resulting graph is the corresponding feasible matching μ. Conversely, if μ is a feasible matching and a buyer buys an object from a seller, then make a link between the buyer and the object. The resulting graph is

two allowable sets of objects S and S', buyer b prefers S to S' at prices p if her total payoff when she buys S is greater than her total payoff when she buys S'. She is indifferent between these two sets if she gets the same total payoff with both sets. Object q is **acceptable** to buyer b at prices p if, at these prices, b likes q at least as well as the null-object.

Under the structure of preferences we are assuming, each buyer b is able to determine which allowable sets of objects she would most prefer to buy at a given price vector p. We denote the set of all such allowable sets by $\boldsymbol{D_b(p)}$ and call it the **demand set of *b* at prices *p*.** (Note that $D_b(p)$ is never empty, because there is always the option of buying $r(b)$ copies of the null object. Note also that, if $S \in D_b(p)$, then every element of S is acceptable to b). This economic structure will be denoted by M^*.

The concepts of corewise-stability and pairwise-stability are usual equilibrium concepts for a variety of cooperative games in the coalitional form.

A feasible payoff x ***is corewise-stable*** *if there is no coalition of players who by forming* ***all*** *their partnerships only among themselves, can all obtain a higher payoff than the one given by x.*

A feasible payoff x is ***pairwise-stable*** *if there are no agents b and q who are not partners, but by becoming partners, possibly dissolving some of their partnerships given by x to remain within their quotas and possibly keeping other ones, can both obtain a higher payoff than the one given by x.*

However, the modern cooperative equilibrium analysis of two-sided matching models relies on a stronger notion than core. The appropriate concept, known as *setwise-stability* (*stability* for short), is the continuous version of the stability concept introduced in Sotomayor (1999-b) for the discrete many-to-many matching model, which in its turn is an extension of the group-stability concept of Roth (1984) for the College Admission model. Setwise-stability captures the idea that instabilities may be caused by coalitions of any size. That is, *a feasible payoff x is* ***setwise-stable*** *if there is no coalition of players who by forming* ***new*** *partnerships only among themselves, possibly dissolving some partnerships of x to remain within their quotas and possibly keeping other ones, can all obtain a higher payoff than the one given by x.*

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the corresponding feasible allocation μ^*.

The coalition in this definition is not required to be able to form **all** their partnerships only among themselves. This makes the difference between the concepts of setwise-stability and corewise-stability and implies that setwise-stable payoffs are in the core. Sotomayor (1999-b) proved that setwise-stability is a new cooperative equilibrium concept for matching models, stronger than pairwise-stability plus corewise-stability. The main point however is that **any stable outcome must be in the core.** For the model we are treating here, Sotomayor (2006) proves that a core payoff may be setwise-unstable. In addition, **an outcome is setwise-stable for M if and only if it is pairwise-stable.**

To formalize the definition of a setwise-stable outcome we need the following notation. Given a feasible payoff (u,w), **$u_b(min)$ is the smallest individual payoff of buyer b; $w_q(min)$ is the smallest individual payoff of seller q.**

Definition 2. *The feasible outcome for M, $(u,w;\mu)$, is* **setwise-stable (stable, for short)** *if $u_b(min) + w_q(min) \geq v_{bq}$ for all pairs (b,q) with $q \notin \mu(b)$.*

If this condition is not satisfied for some pair (b,q), we say that the pair blocks the outcome $(u,w;\mu)$.

The solution concept for market M^* will be called competitive equilibrium outcome. A ***competitive equilibrium outcome*** is a vector of prices for the objects plus a feasible allocation and a corresponding array of payoffs for the buyers, such that each buyer is assigned to an allowable set of objects in her demand set and every unsold object has a zero price.

Definition 3. *The outcome (u,p,μ^*) is a* ***competitive equilibrium outcome*** *for M^* if (i) it is feasible, (ii) μ^* is a feasible allocation such that, if $\mu^*(b)=S$ then $S \in D_b(p)$ for all $b \in B$ and (iii) $p_q=0$ if object q is left unsold.*

If $(u,p;\mu^*)$ is a competitive equilibrium outcome we say that (u,p) is a competitive equilibrium payoff, (p,μ^*) is a ***competitive equilibrium*** *and p is a* ***competitive equilibrium price*** *or an* ***equilibrium price*** *for short.*

If there is an allocation μ^*, compatible with p and satisfying condition (ii) of Definition 2, we say that p is a **competitive price vector.** The allocation μ^* is said to be **compatible** with the competitive price p. The allocation μ^* is called **competitive** if it is compatible with a competitive price. It is proved in Sotomayor (2006) μ is an optimal matching if and only if it is compatible with any competitive equilibrium price vector.

In order to compare two outcomes, agents compare their corresponding total payoffs.

Thus, we can define:

Definition 4. *A stable (respectively, competitive equilibrium) payoff is called a B-optimal stable (respectively, competitive equilibrium) payoff if every player in B weakly prefers it to any other stable (respectively, competitive equilibrium) payoff. That is, the B-optimal stable (respectively, competitive equilibrium) payoff gives to each player in B the highest total payoff among all stable (respectively, competitive equilibrium) payoffs. Similarly we define a Q-optimal stable (respectively, competitive equilibrium) payoff.*

The existence and uniqueness of the B-optimal and of the Q-optimal stable payoffs are proved in Sotomayor (1999-a). The existence and uniqueness of the optimal competitive equilibrium payoffs for each side of the market are proved in Sotomayor (2006). It is also proved in this paper that every seller weakly prefers any competitive equilibrium payoff to the B-optimal competitive equilibrium payoff. Then, if (u^*,v_*) is the B-optimal competitive equilibrium payoff, then v_* is the minimum competitive equilibrium price, in the sense that it is smaller in each component than any other competitive equilibrium price vector. Analogously, if (u_*,v^*) is the Q-optimal competitive equilibrium payoff then v^* is the maximum competitive equilibrium price.

Unlike the cooperative market, **every seller sells all of his items for the same price.** In fact, if a seller has two identical objects, q and q', and $p_q>p_{q'}$ for some price vector p, then no buyer b will demand, at prices p, a set S of objects that contains object q. This is because, by replacing q with q' in S, b gets a more preferable allowable set of objects. But then, q will remain unsold with a positive price, which violates condition *(iii)* of Definition 3.

It must be pointed out that this sort of event does not have anything to do with the quotas of the buyers. It is due to the assumption of the model under which no buyer is interested in acquiring more than one item of a given seller. Without this restriction, if for example a seller sells all of his objects to the same buyer at competitive equilibrium prices, these prices can be distinct.

Remark 1. When every seller sells his identical objects for the same feasible price, we can identify a seller with any of his objects. Thus, we do not cause any confusion by using the same notation for a seller and for any of his objects. Under this observation, if μ^* is a feasible allocation and μ is its corresponding matching, $q \in \mu^*(b)$ means that object q is allocated to buyer b (there is only one object q allocated to buyer b), and $q \in \mu(b)$ means that buyer b and seller q are partners at μ.

On the other hand, since the array of payoffs for any seller q is given by the array of prices of his objects, then, in order to represent the array of the $s(q)$ identical individual payoffs for any seller q, we need not to make any reference to the buyers who are matched to q. For example, $(p_q, p_q, ..., p_q)$ denotes the array of payoffs of seller q and p_q denotes the price of any of his objects.■

According to the exposed above, given an equilibrium price, if a seller does not sell all of his objects at these prices, then any of his objects has zero price. **Therefore, at equilibrium prices, every seller with a positive price will sell all of his items and the number of objects in the market will be enough to meet the demand of all buyers.** However, as it was observed in Sotomayor (2006) if the identical objects are owned by different sellers, they need not to be sold at the same price.

3. DESCRIPTION OF THE MECHANISM

Given a price vector p, the demand set of buyer b may have several elements, say, $C_1,...,C_k$. These elements define k *demand structures for buyer* b, each one of which is given by a collection, $A_b(p)=\{A_{b,1}(p),.. ,A_{b,r(b)}(p)\}$, where the sets $A_{b,i}(p)$ are disjoint and $\cup A_{b,i}(p)=\cup C_i$, $i=1,...,k$. Now, each buyer b is replicated in $r(b)$ copies of herself:

$(b,1),\ldots,(b,r(b))$. Each copy (b,i) demands $A_{b,i}(p)$. We will first give some procedure to define and determine these sets.

First, observe that, given price vector $p\in R^n_+$, the preferences of each buyer $b\in B$ over individual **sellers** can be represented by an ordered list of preferences $L_b(p)$ of the following form:

$L_b(p)= q_1, q_2, q_3, 0, q_4$

indicates that, at prices p, buyer b prefers any object of seller q_1 to any object of seller q_2, any object of seller q_2 to any object of seller q_3, any of these objects to 0, but prefers not to fill her quota to buy any object of seller q_4 at price p. The objects of q_1, q_2 and q_3 are acceptable to b while the ones of q_4 are unacceptable. Buyer b may also be indifferent among several potential partners. Brackets in the preference list will denote this, so, for example, the list

$L_{b'}(p)=[q_1,q_2], q_3,q_4, [q_5,q_6,q_7], 0$

indicates that b' is indifferent between any of the objects of seller q_1 and seller q_2; she prefers any object of these agents to any object of q_3, and so on.

The **bid of b at p, $B_b(p)$,** is a truncation of the preference list $L_b(p)$ so that it contains exactly all $S\in D_b(p)$[10]. To make this clear, consider that $L_b(p)=[q_1,q_2],q_3,q_4,[q_5,q_6,q_7],\ldots$ and b has quota $r(b)=1$. Then, $D_b(p)=\{\{q_1\},\{q_2\}\}$, so $B_b(p)=[q_1,q_2]$; if $r(b)=2$, $D_b(p)=\{\{q_1,q_2\}\}$, so $B_b(p)=[q_1,q_2]$; if $r(b)=4$, $D_b(p)=\{\{q_1,q_2,q_3,q_4\}\}$, so $B_b(p)=[q_1,q_2],q_3,q_4$; if $r(b)=6$, $D_b(p)=\{\{q_1,q_2,q_3,q_4,q_5,q_6\},\{q_1,q_2,q_3,q_4,q_5,q_7\},\{q_1,q_2,q_3,q_4,q_6,q_7\}\}$, so $B_b(p)=[q_1,q_2],q_3,q_4,[q_5,q_6,q_7]$, and so on. Notice that, if $L_b(p)=[q_1,q_2,0], q_3$ and $r(b)=4$, for example, then $D_b(p)=\{\{q_1,q_2,0,0\},\{q_1,0,0,0\},\{q_2,0,0,0\}, \{0,0,0,0\}\}$. In this particular case, we will write $B_b(p)=[q_1,q_2,0,0,0,0]$.

Now, for each non-dummy buyer b, break the ties of $B_b(p)$ in any way desired. The resulting set will be denoted by $B^*_b(p)$. Then, we can define:

$A_{b,i}(p)$ is the set formed by the i-th element in $B^*_b(p)$, if i<r(b);

$A_{b,r(b)}(p)$ is the set formed by the elements in $B^*_b(p)-[A_{b,1}(p)\cup\ldots\cup A_{b,r(b)-1}(p)]$.

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[10] We will say that $q\in B_b(p)$ if and only if q is listed by b in $B_b(p)$. Therefore, $q\in B_b(p)$ if and only if the number of elements of $L_b(p)$ (it may include copies of 0), strictly preferred to q by b at prices p, is less than $r(b)$. Of course, the number of elements listed in $B_b(p)$ (this set may include several copies of the dummy seller), is greater than or equal to $r(b)$.

The set $\mathbf{A_b(p)=\{A_{b,1}(p),..., A_{b,r(b)}(p)\}}$ will be called a **demand structure for b at prices p** (corresponding to the given tie-breaking rule). The set of all $A_{b's}(p)$ will be called a **demand structure at p** and will be denoted by $A(p)$.

To illustrate these definitions, consider that buyer b has quota $r(b)=6$ and that $B_b(p)=[q_1,q_2],q_3,q_4,[q_5,q_6,q_7]$. If we break ties to get $B^*_b(p)=q_1,q_2,q_3,q_4,q_6,q_5,q_7$, the corresponding demand structure for b at prices p is: $A_{b,1}(p)=\{q_1\}$, $A_{b,2}(p)=\{q_2\}$, $A_{b,3}(p)=\{q_3\}$, $A_{b,4}(p)=\{q_4\}$, $A_{b,5}(p)=\{q_6\}$ and $A_{b,6}(p)=\{q_5,q_7\}$. Notice that, if $L_b(p)=[q_1,q_2,0]$, q_3 and $r(b)=3$, for example, then $B_b(p)=[q_1,q_2,0,0,0]$, so we can break ties to get, for example, $B^*_b(p)=0,0,0,q_1,q_2$. In this case, the corresponding demand structure for b at prices p is : $A_{b,1}(p)=\{0\}$, $A_{b,2}(p)=\{0\}$ and $A_{b,3}(p)=\{q_1,q_2,0\}$.

The pair (b,i), with $b\in B$ and $1\leq i\leq r(b)$, is a **loyal demander of S** if $A_{b,i}(p)\subseteq S$.

Definition 5. *Given the feasible price vector $p\in R^n_+$ we will say that the set $S\subseteq Q$ is* ***overdemanded*** *for the demand structure $A(p)$, if there is a set T of loyal demanders of S, such that $|T|>\sum_{q\in S}S(q)$, where $S(q)=\min\{s(q)$, number of $(b,i)\in T$ with $q\in A_{b,i}(p)\}$.*

The overdemanded set S is said to be **minimal,** if no proper subset of S is overdemanded. Thus, for example, if b has quota $r(b)=1$ and $A_{b,1}(p)=\{q_3,q_4\}$; b' has quota $r(b')=2$ and $A_{b',1}(p)=\{q_3\}$ and $A_{b',2}(p)=\{q_4\}$; and b'' has quota $r(b'')=2$ and $A_{b'',1}(p)=\{q_1\}$, $A_{b'',2}(p)=\{q_3\}$, then, $T=\{(b,1),(b',1),(b',2),(b'',2)\}$ is a set of loyal demanders of $S=\{q_3,q_4\}$. If $s(q_3)=1$ and $s(q_4)=3$ then $S(q_3)=\min\{1,3\}=1$ and $S(q_4)=\min\{3,2\}=2$. Then $4=|T|>3=S(q_3)+S(q_4)$. Set S is overdemanded, but it is not minimal. In fact, the set $S'=\{q_3\}$ is overdemanded by $T'=\{(b',1),(b'',2)\}$. Indeed, S' is a minimal overdemanded set.

Remark 2. It follows from the definition of competitive equilibrium that, if $p\in R^n_+$ is a competitive equilibrium price, then each buyer b can be matched to her most preferred allowable set of sellers at prices p (this may include copies of the dummy-seller). Therefore, there is some demand structure $A(p)$ for which each pair (b, i) can be matched to exactly one seller q, with $q\in A_{b,i}(p)$ (q might be the dummy-seller). In addition, every non-dummy seller q is matched $s(q)$ times at most. Hence, by Hall's theorem[11], there is no overdemanded set for $A(p)$.

If $p \in R^n_+$ is not a competitive equilibrium price, then there is no way to match each buyer to her $r(b)$ most preferred sellers under a feasible matching. Again, by Hall's theorem, every demand structure $A(p)$ has an overdemanded set. ■

Now, we can describe the dynamic mechanism. It can be thought of as being an auction procedure. Thus, the matchmaker will be called "auctioneer". We will take all prices and valuations to be integers. The buyers inform their quotas to the auctioneer. Then,

Step (1): The auctioneer announces an initial price vector, $p(1)=(0,...,0)\in R^n_+$. Each buyer b "bids" by announcing $B_b(1)\equiv B_b(p(1))$.

Step (*t*+1): After bids $B_b(t)$ are announced, the auctioneer determines all the demand structures at $p(t)$, using all possible tie-breaking rules. If there is some demand structure $A(t)\equiv A(p(t))$, for which it is possible to match each pair (b, i) to a seller $q\in A_{(b,i)}(t)$, so that no real seller is matched more times than his quota, the algorithm stops. If no such demand structure exists, Hall's Theorem implies that there is some overdemanded set for every demand structure. Then, the auctioneer chooses some demand structure that has the **minimum number of minimal overdemanded sets**, among all demand structures. (This corresponds to the choice of a tie-breaking rule). Next, he selects a minimal overdemanded set for the demand structure chosen and raises the price of all objects belonging to each seller in the set by one unit. All other prices remain at level $p(t)$. This defines $p(t+1)$.

It is clear that the algorithm stops at some step, because, as soon as the price of the objects of a given seller becomes higher than any buyer's valuation for them, the seller will

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[11] Let B and C be two finite disjoint sets. For each b in B, let D_b be a subset of C. A simple assignment is an assignment of C to B, such that each b is assigned exactly one element j of C, such that j is in D_b, and each j in C is assigned to at most one element of B. Then,

THEOREM OF HALL . *A simple assignment exists, if and only if, for every subset B' of B, the number of objects in $D(B')$ is at least as great as the number of buyers in B'.*

See also Gale (1960) for two short proofs of this result.

not be in the bid of any buyer. It follows from the construction of the algorithm, that the final price is a competitive price vector. What is less clear is that this algorithm yields the same price, independent of the demand structures selected by the auctioneer. We will prove this fact in section 4, by showing that the price obtained in the algorithm is the minimum equilibrium price vector. Before, we will illustrate the mechanism with an example.

3.2 EXAMPLE

The following example illustrates the dynamic mechanism. There are four non-dummy-buyers, 1, 2, 3 and 4, and six non-dummy sellers, q_1, q_2, ..., q_6. Seller q_1 has two identical objects and the other sellers have each only one object. The maximum number of objects that each buyer can purchase is given by 3, 2, 1 and 1, respectively. These numbers define the quotas of the buyers. The values of the buyers to the non-null objects are given by the following vectors: v_1=(4,3,3,3,1,1,0), v_2=(2,2,1,0,1,1,0), v_3=(2,0,0,0,0,2,0) and v_4=(1,0,1,1,1,2,0), where the *j*-th coordinate of v_i is the value of any object of seller q_j to buyer *i*.

Step 1. $p(1)$=(0,0,...,0). The matrix of numbers $v_{bq}-p_q(0)$ is given in the table below, with the entries corresponding to the sellers of the demand sets in boldface .

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**4**	**3**	**3**	**3**	1	1	0
2	**2**	**2**	1	0	1	1	0
3	**2**	0	0	0	0	**2**	0
4	1	0	1	1	1	**2**	0

Then, $B_1(1)=q_1,[q_2,q_3,q_4]$; $B_2(1)=[q_1,q_2]$, $B_3(1)=[q_1,q_6]$; $B_4(1)= q_6$. There are three demand structures. The first one is: $A_{1,1}(1)=\{q_1\}$, $A_{1,2}(1)=\{q_2\}$, $A_{1,3}(1)=\{q_3,q_4\}$; $A_{2,1}(1)=\{q_1\}$, $A_{2,2}(1)=\{q_2\}$, $A_{3,1}(1)=\{q_1,q_6\}$ and $A_{4,1}(1)=q_6$. It is not possible to find a competitive matching. There are two minimal overdemanded sets: $\{q_2\}$ and $\{q_1,q_6\}$. The second demand structure is: $A'_{1,1}(1)=\{q_1\}$, $A'_{1,2}(1)=\{q_3\}$, $A'_{1,3}(1)=\{q_2,q_4\}$, $A'_{2,1}(1)=\{q_1\}$, $A'_{2,2}(1)=\{q_2\}$, $A'_{3,1}(1)=\{q_1,q_6\}$ and $A'_{4,1}(1)=q_6$. The only minimal overdemanded set is $\{q_1,q_6\}$. The third demand structure is given by: $A''_{1,1}(1)=\{q_1\}$, $A''_{1,2}(1)=\{q_4\}$, $A''_{1,3}(1)=\{q_2,q_3\}$, $A''_{2,1}(1)=\{q_1\}$, $A''_{2,2}(1)=\{q_2\}$, $A''_{3,1}(1)=\{q_1,q_6\}$ and $A''_{4,1}(1)=q_6$. As before, it is not possible to find a competitive matching. The only minimal overdemanded set is $\{q_1,q_6\}$. The auctioneer must choose a demand structure with the **minimum number of minimal overdemanded sets.** Suppose the auctioneer chooses A'. As a result, he raises the price of all objects of q_1 and q_6 by one unit.

Step 2. $p(2)=(1,0,0,0,0,1,0)$. The matrix of numbers $v_{bq}-p_q(2)$ is given in the table below, with the entries corresponding to the objects of the demand sets in boldface .

	q_1	q_2	q_3	q_4	q_5	q_6	0
1	**3**	**3**	**3**	**3**	1	0	0
2	**1**	**2**	**1**	0	**1**	0	0
3	**1**	0	**0**	**0**	**0**	**1**	**0**
4	**0**	0	**1**	**1**	**1**	**1**	0

Then, $B_1(2)=[q_1,q_2,q_3,q_4]$; $B_2(2)=q_2,\ [q_1,q_3,q_5]$; $B_3(2)=[q_1,q_6]$ and $B_4(2)=[q_3,q_4,q_5,q_6]$. There are several demand structures. Under $A_{1,1}(2)=\{q_1\}$, $A_{1,2}(2)=\{q_2\}$, $A_{1,3}(2)=\{q_3,q_4\}$; $A_{2,1}(2)=\{q_2\}$, $A_{2,2}(2)=\{q_1,q_3,q_5\}$; $A_{3,1}(2)=\{q_1,q_6\}$ and $A_{4,1}(2)=\{q_3,q_4,q_5,q_6\}$, for example, it is not possible to find a competitive matching and the minimal overdemanded set is $\{q_2\}$. However, under $A'_{1,1}(2)=\{q_1\}$, $A'_{1,2}(2)=\{q_3\}$, $A'_{1,3}(2)=\{q_2,q_4\}$ and $A'_{2,1}(2)=\{q_2\}$, $A'_{2,2}(2)=\{q_1,q_3,q_5\}$; $A'_{3,1}(2)=\{q_1,q_6\}$ and $A'_{4,1}(2)=\{q_3,q_4,q_5,q_6\}$, there is a competitive matching that matches buyer 1 to $\{q_1,q_3,q_4\}$, buyer 2 to $\{q_1,q_2\}$, buyer 3 to q_6 and buyer 4 to q_5. Therefore, the final price is $(1,0,0,0,0,1,0)$.■

Remark 3. It is not hard to prove that a set S is minimal overdemanded if and only if the set of all objects of the sellers in S is minimal overdemanded. This is because for all $q\in S$, the number of (b,i)'s, loyal demanders of S with $q\in A_{b,i}(p)$, is strictly greater than $s(q)$, so $S(q)=s(q)$. Therefore, our mechanism is able to operate if we change the vector of prices in R^n by their extension in R^N_{+}.ν

4. MAIN RESULTS

In this section, we first demonstrate that the price vector p produced by the dynamic mechanism is the minimum competitive price. Next we show that the final matching μ can be chosen so that (p,μ) is the minimum competitive equilibrium. As a consequence, the corresponding payoff is the B-optimal stable payoff. Finally, we observe that, by reverting the roles between buyers and sellers in the mechanism, we get the Q-optimal stable payoff. Then we gave a procedure to obtain the Q-optimal competitive equilibrium payoff.

Theorem 1. *Price vector p is the minimum competitive price.*

Proof: Suppose by way of contradiction that p is not the minimum competitive price. Then, there is some competitive price y such that $p\neq y$ and p is not smaller than y. For each step t of the auction denote $U(t)\equiv\{ q\in Q; p_q(t)=y_q\}$. We have that $p(1)=(0,...,0)$, so $p(1)\leq y$. Therefore, since we are working with all integers, there is at least one-step t of the auction such that $U(t)\neq\phi$ and $p(t)\leq y$. From the competitivity of y it follows that there is some demand structure for y with no overdemanded set. Choose one of such demand structures and call it $A^*(y)$.

We need the following lemmas whose proofs are left to the Appendix. Lemma 1 is a technical result, to be used in the proof of Lemma 2. Lemma 2 is a *key lemma*. It implies that if $p_q(t)=y_q$ at some step t of the auction, then the auctioneer will never raise the price of q at any step further. Consequently, $p_q(t)=p_q=y_q$. If this is done, we will have that $p_q\leq y_q$ for all q, because if there was some q such that $p_q>y_q$, then we would have that $p_q(t)=y_q$ at some step t, so q could not have its price raised until the end of the auction, which is a contradiction.

Lemma 1. *Let t be some step of the auction at which $U(t)\neq\phi$ and $p(t)\leq y$. Let $A(t)$ be any demand structure at step t under price $p(t)$. Let $T'=\{(b,i); A_{b,i}(t)\cap U(t)\neq\phi\}$. Suppose that $T'\neq\phi$. Then, there is some demand structure $A'(t)$, such that for each $(b,i)\in T'$, there exists some (b,j), with $A^*_{b,j}(y)\subseteq U(t)$, and such that $A^*_{b,j}(y)=A'_{b,i}(t)$, if $i\neq r(b)$ and $A^*_{b,j}(y)\subseteq A'_{b,i}(t)$, otherwise. Furthermore, $A'_{b,i}(t)= A_{b,i}(t)$ for all $(b,i)\notin T'$.*

Lemma 2. *Let t be some step of the auction at which $U(t)\neq\phi$ and $p(t)\leq y$. Let $A(t)$ be any demand structure at step t under price $p(t)$. Let T' and $A'(t)$ be defined as in Lemma 1. Then, a) $A'(t)$ has no minimal overdemanded set containing elements of $U(t)$; b) every minimal overdemanded set for $A'(t)$, if any, is a minimal overdemanded set for $A(t)$.*

Proof of Theorem 1 *(continued).* Let t be the last step of the auction at which $p(t)\leq y$ and let $S_1=\{q\in Q; p_q(t+1)>y_q\}$. Then, $S_1\neq\phi$. Since we are working with all integers, $S_1\subseteq U(t)$. **Let $A(t)$ be the demand structure chosen by the auctioneer at prices p(t) which has the minimum number of minimal overdemanded sets.** Let S be the minimal overdemanded set for $A(t)$ whose prices are raised at stage $t+1$. Thus, $S=\{q\in Q;$ $p_q(t+1)>p_q(t))\}$, so $S_1= S\cap U(t)$, and so $S\cap U(t)\neq\phi$.

By Lemma 1 and Lemma 2-a, there is some demand structure $A'(t)$, defined from $A(t)$ and $A^*(y)$, that has no minimal overdemanded set containing some element of $U(t)$. Then, S is not a minimal overdemanded set for $A'(t)$. On the other hand, Lemma 2-b asserts that every minimal overdemanded set for $A'(t)$, if any, is a minimal overdemanded set for $A(t)$. Therefore, $A'(t)$ has less minimal overdemanded sets than $A(t)$, contradiction. Hence, p is the minimum competitive price.. ■

Given a competitive price vector *p,* it is not true that there always exists an optimal matching which is compatible to it. (To see this, consider one buyer *b,* two sellers *1* and *2,* every agent with a quota of one and $v_b=(4, 5)$. Price vector $p=(1, 3)$ is competitive. Buyer b demands only the object of seller *1,* which is allocated to her, and the object of seller *2* is unsold. Price p is not a competitive equilibrium price because the price of the unsold object is not zero. We can also observe that the only optimal matching assigns the buyer to seller *2* and this matching is not compatible with *p).* Hence, not always competitive prices are competitive equilibrium prices. However, **this is not the case when the competitive price is the minimum competitive price.**

Theorem 2. *If p is the minimum competitive price then there is a matching μ such that (p, μ) is a competitive equilibrium.*

Proof. Call the objects of seller q *overpriced* if q does not complete his quota under μ but $p_q>0$. Suppose (p, μ) is not a competitive equilibrium, so there is at least one seller k whose objects are overpriced. We will give a procedure for altering μ so as to eliminate the overpriced objects of seller k. For this purpose we construct a direct graph whose vertices are $B \cup Q$. There are two types of arcs. If $q \in \mu(b)$ and b likes q' at least as well as q for all $q' \in \mu(b)$, at prices p, there is an arc from b to q. If q is in $B_b(p)-\mu(b)$ there is an arc from q to b. (Observe that, since every buyer is matched under μ to her favorite set of allowable sellers, it follows that if there is an arc from q to b and an arc from b to q' then b is indifferent between q and q' at prices p). We have that k is in $B_b(p)$ for some $b \notin \mu(k)$, for if not we could decrease p_k a little bit and still have competitive prices, which contradicts the minimality of p. Let $B^* \cup Q^*$ be all vertices that can be reached by a directed path starting from k, followed by $b_1 \notin \mu(k)$.

Case 1: B^* contains a buyer b such that $\mu(b)$ contains the dummy-seller. Then, there is an arc from b to 0. Let $(k=q_1, b_1, q_2, b_2, q_3, \ldots, q_t, b, 0=q_{t+1})$ be a path from k to 0. Then, we may change μ by replacing q_2 by k in $\mu(b_1)$; q_3 by q_2 in $\mu(b_2)$; ..., the dummy-seller q_{t+1} by q_t in $\mu(b)$. Since each b_j is indifferent between q_j and q_{j+1}, for all $j=1,\ldots,t$, the matching is still competitive and k has less one unsold object, and hence he has less one overpriced object.

Case 2: The dummy-seller is not in $\mu(b)$ for every $b \in B^*$. Then, we claim that there must be some q in Q^* such that $p_q=0$, for suppose not. By definition of $B^* \cup Q^*$ we know that if $b \notin B^*$ then $Q^* \cap [B_b(p)- \mu(b)] = \phi$. On the other hand, if $b \in B^*$, $q \notin Q^*$, $q' \in Q^*$ and q and q' are in $\mu(b)$, then b prefers q to q'. Therefore we can decrease the price of the objects of each seller in Q^* by some positive ε and still have competitiveness, contradicting the minimality of p. So choose q in Q^* such that $p_q=0$ and let $(k=q_1, b_1, q_2, b_2, q_3, \ldots, q_t, b_t, q)$ be a path from k to q where $b_1 \notin \mu(k)$. Again change μ by replacing q_2 by k in $\mu(b_1)$, q_3 by q_2 in $\mu(b_2)$, ..., q by q_t in $\mu(b)$ and leaving one object of q unsold. The resulting matching is still competitive. Again the number of unsold objects of k has been reduced and so does the number of overpriced objects.ν

We have proved that the matching obtained in the dynamic mechanism can be chosen so that the resulting allocation is the minimum competitive equilibrium. Then, the outcome produced by the mechanism allocates the sellers to the buyers according to the B-optimal competitive equilibrium payoff. By Proposition 1 below, whose proof can be seen in Sotomayor (2006), the resulting outcome is the B-optimal stable payoff:

Proposition 1. *The payoff (u,p) is the B-optimal stable payoff if and only if it is the B-optimal competitive equilibrium payoff.*

Hence the following corollary is immediate:

Corollary 1. *The outcome produced by the auction mechanism allocates the sellers to the buyers according to the B-optimal stable payoff.*

Proposition 2 below, whose proof is presented in Sotomayor (2006), implies that, under a B-optimal stable outcome no seller discriminates the buyers. Then, the B-optimal stable payoff is competitive. It then follows easily that this outcome coincides with the B-optimal competitive equilibrium outcome.

Proposition 2. *Let (u,w) be the B-optimal (respectively, Q-optimal) stable payoff for M. Let μ be an optimal matching. Then, $w_{bq}=w_{b'q}$ for all $q\in Q$ and all b and b' in $\mu(q)$ (respectively $u_{bq}=u_{bq'}$ for all $b\in B$ and all q and q' in $\mu(b)$).*

Thus, if we change the roles between buyers and sellers in the mechanism **we get the *Q*-optimal stable payoff for the cooperative market *M*.** To obtain the Q-optimal competitive equilibrium payoff use Proposition 3 below, from Sotomayor (2006):

Proposition 3. *Let $(u,w;\mu)$ be a Q-optimal stable outcome. Construct the payoff (u',w') such that the payoffs of a seller q are given by a vector with $s(q)$ repetitions of the number $w'_q=w_q(min)$, and the u'_{bq}'s are given by $u'_{bq}=v_{bq}-w'_q$ if $q\in\mu(b)$. Then, (u',w') is the Q-optimal competitive equilibrium payoff.*

5. MANIPULABILITY QUESTIONS

In this session we will be considering that agents' preferences over potential partners are described by two non negative numbers: For each pair (b, q), a_{bq} and c_{bq} represent, respectively, the minimum payment agent q wants to receive from agent b and the maximum payment b would consider to pay to agent q, if b and q form a partnership. In this case, if s is the transfer from b to q the individual payoffs will be: $u_{bq}=c_{bq} - s$ and $w_{bq}=s - a_{bq}$, respectively. Therefore, $u_{bq} + w_{bq}= c_{bq} - a_{bq}$, so the gain from trade between b and q is given by $v_{bq}= c_{bq} - a_{bq}$ if $c_{bq} \geq a_{bq}$; if $c_{bq} < a_{bq}$ then $v_{bq}=0$, but no possible trade exists between b and q. That is, if q is allocated to b then both agents will have an individual payoff of zero. The market is then given by (B,Q,a,c,r,s), where a is the matrix of numbers a_{bq}'s, c is the matrix of numbers c_{bq}'s and r and s are the vector of quotas $r(b)$'s and $s(q)$'s. For our purposes, it is better to normalize and to work with matrix v . The market (B,Q,v,r,s) is then the many-to-many Assignment game.

We are interested on the manipulability questions that arise when the mechanism that produces the optimal stable payoff for one of the sides is used for a many-to-many Assignment game. Any of the two dynamic mechanisms considered here can be used as an algorithm to implement a strategically equivalent direct mechanism. Thus, we can investigate the manipulability of our dynamic mechanisms by studying the related direct mechanisms. Under the direct mechanism that produces the B-optimal (respectively, Q-optimal) stable outcome, **Q-agents inform numbers a'_{bq}'s and P-agents inform numbers c'_{bq}'s. The other elements of the market are fixed.** The resulting outcome, $(u,w;\mu)$, is the B-optimal (respectively, Q-optimal) stable outcome for the market (B,Q,v',r,s), obtained after normalization. If $(u,w;\mu)$ is produced when agent q misrepresent his/her reservation payments, that is, when $a_{bq}\neq a'_{bq}$ for at least one b, then q's **true individual payoff under $(u,w;\mu)$ is $w'_q=v_{bq}-u_{bq}$ if $\mu(q)\in b$ and $w'_q=0$ if q is unmatched under μ.** Similarly we define the true individual payoff of an agent b.

For the one-to-one Assignment game it is known that (a) any mechanism that produces a stable outcome is manipulable by at least one agent (Theorem 7.3, Roth and Sotomayor, 1990); (b) the allocation mechanism that produces the optimal stable outcome for one side is manipulable by the agents of the opposite side (Demange and Gale, 1985) and (c) the mechanism that produces the optimal stable outcome for one side is strategy-proof for the agents of that side (Demange (1982), Leonard (1981), Demange and Gale (1985)). Therefore, results (a) and (b) apply to the general quota case. As to result (c), it may fail to hold in the general case. In fact, see the following example.

Example 1. Consider the market where $B=\{b_1, b_2\}$ is a set of firms, $Q=\{q_1, q_2, q_3\}$ is a set of workers, $r(b_1)=2$ and $r(b_2)=1$, $s(q_j)=1$ for $j=1,2,3$. The reservation salaries for the workers are zero. The maximum salary b_1 can offer to q_1 is 3, to q_2 is 1 and to q_3 is 2; the maximum salary b_2 can offer to q_1 is 4, to q_2 is 0 and to q_3 is 4. If firms behave straightforwardly, the firm-optimal stable payoff will allocate q_1 and q_2 to b_1 at salaries 1 and 0, respectively; q_3 will be allocate to b_2 at salary 1. Now, let firm b_1 indicate its demands by pretending that the maximum salary it offers to q_2 is 3 instead of 2. The dynamic mechanism that produces the *B*-optimal stable outcome will be as follows:
Step 1: w(1)=(0,0,0). $B_1(1)=q_1,[q_2,q_3]$, $B_2(1)=[q_1,q_3]$. The only demand structure is given by $A_{1,1}(1)=q_1$, $A_{1,2}(1)=\{q_2,q_3\}$, $A_{2,1}(1)=\{q_1,q_3\}$. There is no over-demanded set. The final outcome has the same matching as before, but now every firm pays nothing to the workers. Therefore, straightforward behavior is not a dominant strategy for firm b_1.(Observe that the same outcome could be obtained if b_1 decreased to 1 the maximum salary it offers to q_3).■

For the special case in which the quota of each buyer is the total number of sellers, Theorem 3 shows that the dynamic mechanism that produces the *B*-optimal stable outcome is strategy-proof for the buyers, even when sellers have a quota greater than one[12].

Theorem 3. When *the dynamic mechanism that yields the B-optima (respectively, Q-optimal)l stable payoff is applied to the many-to-many Assignment game, in which the quota of each buyer (respectively, seller) is the total number of sellers (respectively, buyers), then truth telling is a dominant strategy for each buyer (respectively, seller).*

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[12] For a different set up, in which all gross substitute preferences are allowed and one seller owns all the (distinct) objects, Gul and Stacchetti (2000) proves that no ascending price auction mechanism can be efficient and strategy proof (for the bidders).

Proof. We will prove the first assertion. The second one follows dually, by reversing the roles between buyers and sellers. Without loss of generality we can assume that $s(q) \leq |B|$ for all sellers q. In fact, if $s(q)>|B|$ for some q we can include dummy buyers, whose values are zero for all objects, in order to have that $s(q) \leq |B|$ for all q's. When the quota of the buyers is the total number of sellers, a buyer demands some object q whenever its price does not exceed her value to the object. Thus, consider some seller q and denote $k \equiv s(q)$. If $k=|B|$ then $\{q\}$ will not be over-demanded at zero price and every buyer will buy an object of seller q at zero price. Then, consider $k<|B|$. If the price of the objects of seller q is less than its $k+1$-th highest value, there will be at least $k+1$ copies of bidders demanding only q. Then $\{q\}$ will be over-demanded at any demand structure. As the price of the objects of q reaches its $k+1$-th highest value, there will be some demand structure such that, k copies of bidders at most will demand only q (these bidders have the k-th highest values for the objects of seller q) and any other copy of bidders that demands q will also demand the null object. Thus, at this price, q will not belong to any over-demanded set and its final price will be its $k+1$-th highest value. Therefore, each seller q will sell his set of $s(q)$ objects to the set of k bidders who value them most and at the price of its $k+1$-th highest value. Thus, the same outcome can be obtained if each seller q sells his objects independently through the corresponding Vickrey's auction of $s(q)$ identical objects with unitary demands. This implies that the compatibility of incentives observed in that auction applies and we have completed the proof. ■

It is more natural to think of workers having different reservation salaries for different firms, than sellers having different reservation prices for different buyers. Then we will think of agents as being firms and workers. Our main finding is that the many-to-one Assignment game (workers have a quota of one) is the most general Assignment game for which the dynamic mechanism that yields the worker optimal stable outcome is strategy-proof for the workers. In fact, this follows from Example 1 and Theorem 4 below. Theorem 4 implies that, for the special case in which all workers have quota of one, no coalition of workers can, by misrepresenting their reservation salaries, obtain a true outcome which is preferred to the Q-optimal stable payoff for the original market, by all its members. (Of course, an analogous result applies for the firms, if they have quota of one).

Theorem 4. *Suppose all workers have quota of one. Let* Q' *be the set of workers who misrepresent their reservation salaries. Let* $(u,w;\mu)$ *be any stable outcome for the market* (B,Q,v'), *where* v' *agrees with* v *on* $B \times (Q-Q')$. *Let* (u_*,w^*,μ^*) *be the Q-optimal stable outcome for* (B,Q,v,r) *and let* $(u,w';\mu)$ *be the true outcome under* $(u,w;\mu)$. *Then,* $w^*_q \geq w'_q$ *for at least one* q *in* Q'.

For the proof we need the lemmas below. We will be denoting any of the individual payoffs of a firm b under the Q-optimal stable payoff (u_*,w^*) by u_{*b}. This does not cause any confusion because a firm has equal individual payoffs under (u_*,w^*). (Proposition 2).

Lemma 3. *Let* $(u_*,w^*;\mu^*)$ *be a* Q*-optimal stable outcome. If* $\sum_{q\in Q} s(q) \leq \sum_{b\in B} r(b)$. *then* $u_{*b}=0$ *for some* b *in* B.

Proof. The proof is immediate unless $\sum_{q\in Q} s(q)= \sum_{b\in B} r(b)$ and all of Q complete their quota under $(u_*,w^*;\mu^*)$. Suppose that even in this case $u_{*b}>0$ for all b in B. Then, there exists some $\lambda>0$ such that, by decreasing of λ the individual payoffs of every firm and by increasing of λ the individual payoffs of every worker we still get a stable outcome, which contradicts the Q-maximality of w^*. ■

Lemma 4. *Let* $(u_*,w^*;\mu^*)$ *be a* Q*-optimal stable outcome. Suppose* $u_{*b}>0$ *for all* $b\in B' \subseteq B$ *and let* $Q'=\{q;\ \mu^*(q)\subseteq B'\}$. *Then there is a pair* (b,q) *with* $u_{*b} + w^*_q(min)= v_{bq}$, *and* $b\in B'$, $q\in Q-Q'$ *and* $b\notin\mu^*(q)$.

Proof. Note that from Lemma 3, $Q-Q'\neq\phi$. Arguing by contradiction, suppose $u_{*b}+w^*_q(min)>v_{bq}$ for all $b\in B'$, $q\in Q-Q'$ *and* $b\notin\mu^*(q)$. Then for some positive λ, $u_{*b}-\lambda>0$ and

$$(u_{*b}-\lambda)+w^*_q(min)>v_{bq}, \qquad (1)$$

Now define $(u',w';\mu^*)$ as follows:

$u'_b= u_{*b}-\lambda$ for $b\in B'$

$\quad =u_{*b}$ for $b\in B-B'$;

$w'_{bq} = w^*_{bq}+\lambda,$ if $b\in\mu^*(q)$ and $b\in B'$

$=w^*_{bq}$ if $b\in\mu^*(q)$ and $b\in B\text{-}B'$.

We claim that this outcome is stable. In fact, the only possible unstable pairs $(b.q)$ must have either $b\in B'$, $q\in Q\text{-}Q'$ and $b\notin\mu^*(q)$ or $b\in B\text{-}B'$, $q\in\mu^*(B')$ and $b\notin\mu^*(q)$. In the first case it is not possible because of (1). In the second case, $u'_b+w'_q(min)\geq u_{*b}+w^*_q(min)\geq v_{bq}$, by stability of (u_*,w^*). However, $w^* < w'$, which contradicts the Q-maximality of w^*. ■

Lemma 5. Suppose *all workers have quota of one. Let* $(u_*,w^*;\mu^*)$ *be a* Q*-optimal stable outcome. Let* $(u,w;\ \mu)$ *be a feasible outcome and suppose* $Q^+\equiv\{q\in Q;\ w_q>w^*_q\}$ *is non-empty. Then there is a pair* (b,q), $b\in\mu\ (Q^+)$, $b\notin\mu(q)$ *and* $q\notin Q^+$ *such that* $u_b(min) + w_q<v_{bq}$.

Proof. Let $B^+=\mu\ (Q^+)$. There are two cases.

Case 1. There is some $b\in B^+$ such that $b\notin\mu^*(Q^+)$. Let $k\in Q^+$ such that $b\in\mu\ (k)$. Then, since $w_k>w^*_k$ we must have that $u_{bk}<u_{*b}$ for if not, $v_{bk}= u_{bk}+w_k >u_{*b} + w^*_k$, which contradicts stability of (u_*,w^*). Hence $u_{*b}>0$, so b completes its quota under μ^* and with workers not in Q^+. Let $q\in\mu^*(b)$. Then, $w_q\leq w^*_q$. It follows that, $u_b(min) + w_q\leq u_{bk} + w_q < u_{*b} + w^*_q =v_{bq}$. Hence the assertion is proved.

Case 2. $B^+=\mu\ (Q^+)\subseteq\mu^*(Q^+)$. The fact that $w_q>0$ for all $q\in Q^+$ implies that all agents in Q^+ are matched under μ, so $B^+=\mu^*\ (Q^+)$. Let $b\in B^+$. Then, $b=\mu\ (k)$ for some $k\in Q^+$. Following case 1 we have that $u_{bk}<u_{*b}$. Then, $u_{*b}>0$ and $u_b(min)<u_{*b}$ for all $b\in B^+$. By Lemma 4, there is a pair (b,q) with $u_{*b} + w^*_q= v_{bq}$, and $b\in B^+$ and $q\in Q\text{-}Q^+$. The result follows since $w_q\leq w^*_q$ and $u_b(min)<u_{*b}$. ■

Proof of Theorem 4. Suppose all workers in Q' are better off at $(u,w;\mu)$ than at $(u_*,w^*;\mu^*)$, under their true reservation payoffs . Let $Q^+=\{q\in Q;\ w'_q>w^*_q\}$. Then, $Q'\subseteq Q^+$ and so $Q^+\neq\phi$. The outcome $(u,w';\mu)$ is clearly feasible for the original market (observe that $w'_q>0$ for every q in Q'). By Lemma 5 there exists a pair (b,q), $b\in\mu\ (Q^+)$ and $q\notin Q^+$ such that $u_b(min) + w'_q<v_{bq}$. This means that $q\notin Q'$ and hence $w'_q=w_q$ and

$v'_{bq}=v_{bq}$. But then, $u_b(min) + w_q < v'_{bq}$, which contradicts the stability of $(u,w;\mu)$ in (B,Q,v',r,s). ■

An immediate consequence of Theorem 4 is:

Corollary 2. *Suppose all workers have quota of one and the direct mechanism yields the optimal stable outcome for the workers. Then, it is a dominant strategy for each worker to state his/her true reservation salary.*

6. FINAL REMARKS AND RELATED WORK

We considered the many-to-many Assignment game introduced in Sotomayor (1992). In Sotomayor (2006), the relationship between the optimal stable payoffs and the corresponding optimal competitive equilibrium payoffs for this model is established: The *B*-optimal stable payoff equals the *B*-optimal competitive equilibrium payoff, but the *Q*-. optimal competitive equilibrium payoff may be different from the *Q*-optimal stable payoff. The *Q*-optimal competitive equilibrium payoff can be obtained from the *Q*-optimal stable payoff by reducing the price of each of the items of every seller to his minimal individual payoff.

In the present paper we provided a dynamic mechanism to finding such optimal stable outcomes. We proved that the mechanism operates in a finite number of steps, and converges to the extreme point of the lattice of stable payoffs favored by players on the offer-making side of the market. The two extreme points of the lattice of competitive equilibrium payoffs can then be obtained via the application of the result of Sotomayor (2006) mentioned above.

In order to get the most preferred outcome by the buyers, the mechanism keeps the transfer to the sellers as low as possible without causing excess demand. Since all prices and values are integers, a transfer is made to a seller by having the prices of all his objects equally increased by one unit. Then, at the end of the mechanism, the identical objects of a seller are sold for the same price. To which sellers should the transfers be made in each step of the algorithm was the main issue we faced in designing the algorithm presented here.

Proving that these transfers lead to the minimum competitive price was the hardest task. The difficulty was caused by the fact that agents may form more than one partnership.

A virtue of the dynamic mechanism presented here is that its ascending bids imitate a real auction mechanism: as its final step approaches, each buyer is able to know which set of objects she is more likely to obtain and, roughly, how much she will have to pay for it. On the other hand, at each stage of the mechanism, all objects are open for bidding simultaneously. Thus, a buyer is able to select whatever set of objects she wishes, as well as switch some of the items for others, if these items become too expensive.

The mechanism presented here implements a direct mechanism, where buyers state their valuations and sellers state their reservation prices. Some results concerning the manipulability of this direct mechanism were obtained. We showed that, unlike the quota one case, our mechanism is manipulable by the players from both sides. When every buyer (respectively, seller) has total quota, the mechanism that produces the buyer- optimal (respectively, seller-optimal) stable payoff is strategy-proof for the buyers (respectively, sellers). Analogous results to the ones obtained for the College Admission model of Gale and Shapley (see Roth and Sotomayor, 1990) were proved for the many-to-one Assignment game. Namely, when every buyer (respectively, seller) has a quota of one, the mechanism that yields the B-optimal (respectively, Q-optimal) stable payoff is manipulable by the sellers (respectively, buyers) and it is not individually manipulable by the buyers (respectively, sellers). Also, no coalition of buyers (respectively, sellers) can, by falsifying demands, get a true outcome which is better than any stable outcome under the true valuations for all its members. This way, the many-to-one Assignment game where buyers (respectively, sellers) have a quota of one can be considered as the most general Assignment game for which the dynamic mechanism that yields the buyer-optimal (respectively, seller-optimal) stable outcome is strategy-proof for the buyers (respectively, sellers).

The direct mechanism that yields the worker-optimal stable outcome can be used as a centralized procedure in some large labor markets in the entrance level. A market that would fit into this scenery might be a version of the resident-hospital market in the United States with money. The entry-level labor market for medical residents and hospitals in the United States has been modeled in the literature as the College Admission model of Gale and Shapley. The salaries that hospitals offer to the 20,000 residents, each year, are not negotiated. However, if hospitals eventually change their policy by allowing monetary negotiation with the residents, the actual allocation mechanism will no longer be appropriated[13]. In fact, the algorithm employed by the National Resident Matching Program, which is a version of the deferred-acceptance algorithm of Gale and Shapley, does not contemplate the possibility of salary negotiation.

The evidence has shown that allocation procedures that elicit a high level of participation use simple rules, which are easy to be followed and understood by every agent. Nevertheless, the literature does not provide an allocation mechanism with simple rules, which can be used, in practice, for such large markets. An appropriate allocation mechanism would require that agents were able to provide a sophisticated level of information. If, for instance, the set of permissible salaries is discrete, agents have preferences over all pairs made up with a potential partner and a permissible salary. Under a centralized procedure, the participants should inform the matchmaker these preferences and an algorithm would find the allocation. However, it is almost always unfeasible for an agent to construct an ordered list of preferences over all such a pairs (consider, for example, that the set of acceptable hospitals to a given resident has ten hospitals and all hospitals have the same set of permissible salaries, with only three elements. The resident faces the problem of ordering thirty pairs!). If it is assumed that the salaries vary continuously in the set of non-negative real numbers, then the lists of preference would be replaced by continuous utility functions. Of course, a rule that states that every agent must inform all such utility functions is not easy to be followed by the agents of this market, unless these functions are linear.

Although it is not our intention to solve the problem of finding an appropriate allocation mechanism for the resident-hospital market with money in this paper, there are, under our point of view, potential applications of our mechanism to this kind of large market. In this event, our mechanism gives a satisfactory solution in almost all respects aside its reliance on its strong linearity of the utility functions assumption.

[13] Recently, the NRMP was involved in a lawsuit about resident stipends and duty hours. Residents complained against the fact that the algorithm used by the NRMP does not contemplate the possibility of salary negotiation and enables hospitals to underpay residents. (Jung et al. Versus Association of American Medical Colleges et al., 02-CV-00873 (DDC 2002); Chae (2003), Miller and Greaney (2003); see also http://www.savethematch.org and http://www.savetheresidents.com/index.asp).

Since every worker has a quota of one, it is very simple to find a minimal over-demanded set when the mechanism produces the worker-optimal stable payoff. There is only one demand structure in each step of the algorithm. Then *a set* S *of firms' positions is over-demanded if there is a set* T *of workers, loyal demanders of* S*, such that* $|T|>\sum_{b\in S} S(b)$*, where* $S(b)$ *is the minimum between b's quota and the number of workers in* T *who demand* h*.*

The algorithm with the workers making offers resembles an auction of the positions offered by the firms:

Step (1): The auctioneer announces an initial payoff vector, $u(1)= (0,...,0)\in R^n_+$. Each worker q informs his/her demand set $D_q(1)$.

Step (*t*+1): After the demand sets $D_q(t)$'s are announced, if it is possible to match each worker q to a firm $b\in D_q(t)$, so that no real worker is matched twice, the algorithm stops. If no such demand structure exists, the auctioneer chooses some **minimal over-demanded set** and raises the payoff corresponding to all positions being offered by each firm in the set by one unit. All other payoffs remain at level $u(t)$. This defines $u(t+1)$.

According to our previous results applied to this many-to-one Assignment game, as it happens in the deferred-acceptance algorithm of Gale and Shapley, no worker will be tempted to misrepresent his/her reservation salary. Also, as in the discrete many-to-one case, if the algorithm produces the optimal stable payoff for the firms, then the mechanism is manipulable by the firms.

Competitive equilibria have been used by several authors, besides those cited along this paper, to produce allocations with desirable properties of fairness and efficiency. There is by now a vast theoretic literature about two-sided matching markets, providing mechanisms to produce such allocations. For the buyer-seller market game proposed by Shapley and Shubik, Sotomayor (2002) proposes a new descending bid method for auctioning multiple objects, which generalizes the Dutch auction and produces the maximum competitive equilibrium price. Demange, Gale and Sotomayor (1986) also consider another auction mechanism which approximates the minimum competitive equilibrium price to any desired degree of accuracy. For the case in which the utility functions are piecewise linear, Alkan (1988) presents a dynamic mechanism that finds an equilibrium price in finitely many steps and approximates an equilibrium price for general continuous utilities. Sotomayor (1992) presents a procedure to obtain the optimal stable payoffs of the many-to-many case which consists in solving three linear programming problems.

APPENDIX

In this section we will demonstrate some of the results stated in section 4. The following remark will be useful for the proofs.

Remark 4. Let t be some step of the auction at which the set $U(t)\neq\phi$ and $p(t)\leq y$. Let $A(t)$ be any demand structure at step t. Write the set of elements of $\mathbf{B_b(t)}$ as $\mathbf{F(t)\cup G(t)}$ and the set of elements of $\mathbf{B_b(y)}$ as $\mathbf{F(y)\cup G(y)}$, with $G(t)\neq\phi$, $G(y)\neq\phi$, $F(t)\cap G(t)=\phi$, $F(y)\cap G(y)=\phi$, such that b is indifferent between any two sellers of $G(t)$ (respectively $G(y)$) at price $p(t)$ (respectively y). In addition, b strictly prefers any object of $F(t)$ (respectively $F(y)$), if any, to any object of $G(t)$ (respectively $G(y)$) at price $p(t)$ (respectively y) (Observe that $G(t)$ as well as $G(y)$ may have only one element). It is clear that, for all $j\neq r(b)$, **either $A_{b,j}(t)\subseteq F(t)$ (respectively $A^*_{b,j}(y)\subseteq F(y)$) or $A_{b,j}(t)\subseteq G(t)$ (respectively $A^*_{b,j}(y)\subseteq G(y)$). Also, $A_{b,r(b)}(t)\subseteq G(t)$ (respectively, $A^*_{b,r(b)}(y)\subseteq G(y)$) and if $A_{b,j}(t)\subseteq F(t)$ (respectively $A^*_{b,j}(y)\subseteq F(y)$) then $|A_{b,j}(t)|=1$ (respectively $|A^*_{b,j}(y)|=1$).**

Now, let $C\subseteq Q$. Suppose that $p_q(t)=y_q$ for all $q\in C$, b is indifferent between any two elements of C at prices $p(t)$ (and so is at prices y), $\mathbf{C\subseteq G(t)}$ **and** $\mathbf{C\subseteq G(y)}$. Then, $F(t)$ (respectively $F(y)$) is the set of all objects that are strictly preferred by b to any object of C at price $p(t)$ (respectively y). Also, $F(t)\cup G(t)$ (respectively $F(y)\cup G(y)$) is the set of all objects that are weakly preferred by b to any object of C at price $p(t)$ (respectively y). It can be shown that

$$\mathbf{F(y)\cup G(y) \subseteq F(t)\cup G(t) \text{ and } F(y)\subseteq F(t)} \qquad \textbf{(A.1)}$$

$$\mathbf{G(y)\cap U(t)=G(t)\cap U(t) \text{ and } F(y)\cap U(t)=F(t)\cap U(t).} \qquad \textbf{(A.2)}$$

Proof of (A.1). To prove the first inclusion, if $q''\in \mathbf{F(y)\cup G(y)}$, then $v_{bq''} - y_{q''} \geq v_{bq}-y_q \;\; \forall q\in C$. Using that $p_q(t)=y_q \;\; \forall q\in C$ and $y_{q''} \geq p_{q''}(t)$, we get that $v_{bq''} - p_{q''}(t) \geq v_{bq''}-y_{q''} \geq v_{bq}-y_q = v_{bq}-p_q(t) \;\; \forall q\in C$, so $q''\in \mathbf{F(t)\cup G(t)}$. Hence, $\mathbf{F(y)\cup G(y) \subseteq F(t)\cup G(t)}$. The proof of the other inclusion is analogous.

Proof of (A.2). Observe that, if $q\in G(y)\cap U(t)$, then the facts that $p_q(t)=y_q$, b is indifferent between q and any element of C at price y, and $p_{q'}(t)=y_{q'}$ for all $q'\in C$, imply that $v_{bq}-p_q(t)= v_{bq}-y_q= v_{bq'} - y_{q'} = v_{bq'} - p_{q'}(t)$, so b is indifferent between q and any element of C at price $p(t)$, so $q\in G(t)$, so

$G(y) \cap U(t) \subseteq G(t) \cap U(t)$. With an analogous argument we prove that $G(t) \cap U(t) \subseteq G(y) \cap U(t)$. We also have that $F(y) \cap U(t) \subseteq F(t) \cap U(t)$, by (A.1). Now, if $q \in F(t) \cap U(t)$ then q is strictly preferred by b to any object of C at price $p(t)$ (because $C \subseteq G(t)$). Since $p_q(t)=y_q$, then $v_{bq}-y_q = v_{bq}-p_q(t) > v_{bq'}-p_{q'}(t) = v_{bq'} - y_{q'}$ for all $q' \in C$, so b strictly prefers q to any element of C at prices y, so $q \in F(y)$ and so $F(t) \cap U(t) \subseteq F(y) \cap U(t)$.

It is also important to point out that,

if $A_{b,i}(t) \subseteq F(t) \cap U(t)$, for some (b,i), then $A_{b,i}(t) = A^*_{b,j}(y)$, for some (b,j). (A.3)

In fact, suppose that $A_{b,i}(t) \subseteq F(t) \cap U(t)$, for some (b,i). Then, $A_{b,i}(t) = \{q\}$ for some q and $q \in F(y) \cap U(t)$, by (A.2), so $A^*_{b,j}(y) = \{q\}$ for some (b,j). ■

The original proof of Lemma 1 is very technical and long. The details of this proof are with the author, available to the interested readers. We present below a sketch of the proof that shows the main steps of it.

Sketch of the proof of Lemma 1. Define $A'(t)$ as follows. If $(b,i) \notin T'$, set $A'_{b,i}(t) \equiv A_{b,i}(t)$. If for all $(b,i) \in T'$ there is some (b,j), such that $A^*_{b,j}(y) \subseteq A_{b,i}(t) \cap U(t)$, define $A'_{b,i}(t) \equiv A_{b,i}(t)$ for all $(b,i) \in T'$ and we are done. Otherwise, there is some $(b,i) \in T'$, with $A_{b,i}(t) \equiv C \cup E$, where $C = A_{b,i}(t) \cap U(t)$, such that,

for all (b,j), $A^*_{b,j}(y)$ is not contained in C. (1)

We want to show that it is possible to define $A'(t)$ so that, for all $(b,j) \in T'$, with $j \neq r(b)$, there exists some (b,k) such that $A'_{b,j}(t) = A^*_{b,k}(y) \subseteq U(t)$; if $(b,r(b)) \in T'$, there exists some (b,k) such that $A^*_{b,k}(y) \subseteq A'_{b,r(b)}(t) \cap U(t)$.

The plan of the proof is the following: By defining $F(t)$, $G(t)$, $F(y)$ and $G(y)$ as in Remark 5, we first show that $\boldsymbol{C \subseteq G(t)}$, $C \subseteq B_b(y)$ and $\boldsymbol{C \subseteq G(y)}$, so we have satisfied all the hypothesis of Remark 5 and so (A.1) and (A.2) hold. From $C \subseteq B_b(y)$ it follows that all of C must be demanded by b at prices y, so every element of C must be in some $A^*_{b,j}(y)$ for some copy (b,j). Then, by (1), we conclude that such a copy of b is $(b,r(b))$. Then, $A^*_{b,r(b)}(y) = C \cup D$, with $D \cap C = \phi$ and $D \neq \phi$ by (1). It is clear that $A^*_{b,r(b)}(y) \subseteq G(y)$, because $(b,r(b))$ is the last copy of b. Now set:

$\Gamma \equiv \{ (b,j); A_{b,j}(t) \subseteq G(t)$ and $A_{b,j}(t) \cap U(t) \neq \phi \}$

$\Gamma' \equiv \{ (b,j); A^*_{b,j}(y) \subseteq G(y)$ and $A^*_{b,j}(y) \cap U(t) \neq \phi \}$

$\Im \equiv \{ (b,j); A_{b,j}(t) \subseteq G(t)$ and $A_{b,j}(t) \cap U(t) = \phi \}$

$\Im' \equiv \{ (b,j); A^*_{b,j}(y) \subseteq G(y)$ and $A^*_{b,j}(y) \cap U(t) = \phi \}$

We have that $\Gamma \neq \phi$, since $(b,i) \in \Gamma$. Also, $\Gamma' \neq \phi$, since $(b,r(b)) \in \Gamma'$.

The next step is to define a one-to-one map f from $\Gamma - \{(b,r(b)\}$ into $\Gamma' - \{(b,r(b)\}$. This can be done by establishing that $|\Gamma| \leq |\Gamma'|$ and $(b,r(b)) \in \Gamma$. Then, define

$A'_{b,j}(t) \equiv A_{b,j}(t)$ if $A_{b,j}(t) \subseteq F(t)$ or $(b,j) \in \Im$.

$A'_{b,j}(t) \equiv A^*_{f(b,j)}(y)$ if $(b,j) \in \Gamma - \{(b,r(b))\}$.

$$\boldsymbol{A'_{b,r(b)}(t) \equiv G(t) - \bigcup_{j\neq r(b)} A'_{b,j}(t).}$$

To see that $A'(t)$ is well defined and is the desired demand structure, use (A.1) and (A.2) of Remark 5.ν

Proof of Lemma 2. For part a), suppose by way of contradiction that S is a minimal overdemanded set for $A'(t)$ and $S_1\equiv S\cap U(t)\neq\phi$. Let T be the set of loyal demanders of S. The fact that S is overdemanded means exactly that

$$|T| > \sum_{q\in S} S(q)| \qquad \textbf{(1)}$$

We will show that $S\text{-}S_1$ is non-empty and overdemanded for $A'(t)$, so S is not a minimal overdemanded set for $A'(t)$, which is a contradiction. To see this, define $T_1=\{(b,i)\in T;\ A'_{b,i}(t)\cap S_1\neq\phi\}$. Let T' be as defined in Lemma 1. Now, observe that $T_1\subseteq T'$. In fact, if $(b,i)\notin T'$ then $A_{b,i}(t)=A'_{b,i}(t)$, so $A'_{b,i}(t)\cap U(t)=\phi$, and so $(b,i)\notin T_1$. By Lemma 1, for each $(b,i)\in T_1$ there is some (b,j), such that $A^*_{b,j}(y)\subseteq A'_{b,i}(t)\cap U(t)$. On the other hand, the fact that $(b,i)\in T$ implies that $A'_{b,i}(t)\subseteq S$, so $A'_{b,i}(t)\cap U(t) = A'_{b,i}(t)\cap S_1$. Then, $A^*_{b,j}(y)\subseteq A'_{b,i}(t)\cap S_1$, so $A^*_{b,j}(y)\subseteq S_1$. Thus, since $A'_{b,i}(t)\cap A'_{b,k}(t)=\phi$ if $i\neq k$,

$$\boldsymbol{|T_1|\leq|\{(b,j);\ A^*_{b,j}(y)\subseteq S_1\}|\leq} \sum_{q\in S_1} S_1(q)\,, \qquad \textbf{(2)}$$

where the last inequality is due to the competitivity of y. But then, (1) and (2) imply that $|T\text{-}T_1|=|T|\text{-}|T_1|>\sum_{q\in S} S(q) - \sum_{q\in S_1} S_1(q) = \sum_{q\in S-S_1} S(q) \geq 0$, from which follows that $T\text{-}T_1\neq\phi$. However, $T\text{-}T_1=\{(b,i)\in T;\ A'_{b,i}(t)\subseteq S\text{-}S_1\}$, so $S\text{-}S_1$ is non-empty and overdemanded for $A'(t)$, as we wanted to show.

For part b), suppose that $A'(t)$ has overdemanded sets. Let S be some minimal overdemanded set for $A'(t)$. Let T be the set of loyal demanders of S. Let T' be as defined in Lemma 1. By part a), $S\cap U(t)=\phi$. Then, if $(b,i)\in T$, $A'_{b,i}(t)\subseteq S$, so $A'_{b,i}(t)\cap U(t)=\phi$, so $(b,i)\notin T'$. By Lemma 1, $A'_{b,i}(t)=A_{b,i}(t)$. Then, for all $(b,i)\in T$, $A_{b,i}(t)\subseteq S$. Then, T is a set of loyal demanders of S under $A(t)$ and $min\{s(q)$, number of $(b,i)\in T$ with $q\in A_{b,i}(t)\} = min\{s(q)$, number of $(b,i)\in T$ with $q\in A'_{b,i}(t)\}$. Hence, S is also minimal overdemanded for $A(t)$, and the proof is complete. ■

REFERENCES

Alkan, A. (1988) "Auctioning several objects simultaneously". Bogazici University, *mimeo.*

Chae, S. H. (2003) "Is the match illegal?," *The New England Journal of Medecine* 348, 352-356.

Crawford, V. P. and Elsie M. Knoer (1981) "Job matching with heterogeneous firms

and workers”, *Econometrica* 49, n.2, 437-450.
Dantzig, G. B. (1963) “Linear Programming and Extensions”, Princeton University Press.
Demange G. (1982) “Strategyproofness in the assignment market game”, Preprint. Paris: Ecole Polytechnique, Laboratoire D’Econometrie.
Demange, G., David Gale and Marilda Sotomayor (1986) “Multi-item auctions”, *Journal of Political Economy* vol 94, n.4, 863-872.
Gale, D. (1960) “The theory of linear economic models”, New York: McGraw-Hill.
Gale, D. and L. Shapley (1962) "College admissions and the stability of marriage", *American Mathematical Monthly*, 69, 9-15.
Gul, F. and E. Stacchetti (2000) “The english auction with differentiated commodities”, *Journal of Economic Theory* 92, 66-95.
Hall P., (1935) “On representatives of subsets”, *J. London Math. Soc.* 10, 26-30.
Kelso, A. and Vincent P. Crawford (1982) “Job matching, coalition formation, and gross substitutes”, *Econometrica* 50, n.6, 1483-1504.
Leonard, H. B. (1983) “Elicitation of honest preferences for the assignment of individuals to positions”, *Journal of Political Economy* 461-479.
Miller, F. H. And Thomas L. Greaney (2003) “The National Resident Matching Program and Antitrust Law”, *Journal of the American Medical Association* 289, 913-918.
Roth A. and Marilda Sotomayor (1990) “Two-sided matching. A study in game-theoretic modeling and analysis”, *Econometric Society Monograph Series, N. 18 Cambridge University Press.*
Shapley, L. and Martin Shubik (1972) “The assignment game I: The core”, *International Journal of Game Theory,* 1, 111-130.
Sotomayor, M. (1992) “The multiple partners game”, *Equilibrium and Dynamics: Essays in Honor of David Gale,* edit. by Mukul Majumdar, The Macmillan Press Ltd.
_____________(1999) “The lattice structure of the set of stable outcomes of the multiple partners assignment game”, *International Journal of Game Theory* 28, 567-583.
____________(2002) "A simultaneous descending bid auction for multiple items and unitary demand", *Brazilian Economic Journal* 56(3), 497-510.

_____________(2006) "Connecting the cooperative and competitive structures of the multiple-partners assignment game", *Journal of Economics Theory,* to appear.

Vickrey, William (1961) "Counterspeculation, auctions, and competitive sealed tenders" *Journal of Finance* 16, 8-37.

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Detailed proof of Lemma 1

Proof of Lemma 1. Define $A'(t)$ as follows. If $(b,i)\notin T'$, set $A'_{b,i}(t)\equiv A_{b,i}(t)$. If for all $(b,i)\in T'$ there is some (b,j), such that $A^*_{b,j}(y)\subseteq A_{b,i}(t)\cap U(t)$, define $A'_{b,i}(t)\equiv A_{b,i}(t)$ for all $(b,i)\in T'$ and we are done. Otherwise, there is some $(b,i)\in T'$ such that,

for all (b,j), $A^*_{b,j}(y)$ is not contained in $A_{b,i}(t)\cap U(t)$. (1)

We want to show that it is possible to define $A'(t)$ so that, for all $(b,j)\,)\in T'$, with $j\neq r(b)$, there exists some (b,k) such that $A'_{b,j}(t)=A^*_{b,k}(y)\subseteq U(t)$; if $(b,r(b))\in T'$, there exists some (b,k) such that $A^*_{b,k}(y)\subseteq A'_{b,r(b)}(t)\cap U(t)$.

Set $A_{b,i}(t)\equiv C\cup E$, where $C= A_{b,i}(t)\cap U(t)$ and $C\cap E=\phi$. We have that $C\neq\phi$, due to the fact that $(b,i)\in$T'. Define $F(t)$, $G(t)$, $F(y)$ and $G(y)$ as in Remark 5. Then, C is contained in the set of elements listed in $B_b(t)$. We claim that

$$\mathbf{A_{b,i}(t)\subseteq G(t).} \qquad \textbf{(2)}$$

In fact, if $A_{b,i}(t)\subseteq F(t)$, then $|A_{b,i}(t)|=1$, and so $A_{b,i}(t)=\{q\}$, for some $q\in U(t)$. We do not have that $q\in F(y)$, for if not there would be some (b,j) such that $A^*_{b,j}(y)=\{q\}= A_{b,i}(t)$, which contradicts (1). Then suppose $q\notin F(y)$. We are going to show that $F(y)\cup G(y)\subseteq F(t)$, so $|F(y)\cup G(y)|\leq|F(t)|<$r(b) (use that $G(t)\neq\phi$, so $F(t)$ is the set of objects which are strictly preferred by b to any object of $G(t)$. Now use Remark 2). But this is absurd since $|B_b(y)|\geq r(b)$. Then, take any $q'\in G(y)$. If b prefers q to q' at $p(t)$ then $v_{bq}-y_q = v_{bq}-p_q(t) > v_{bq'} - p_{q'}(t) \geq v_{bq'} -y_{q'}$, so $v_{bq}-y_q > v_{bq'} - y_{q'}$, from which follows that $q\in F(y)$, contradiction. Then, b prefers q' to q or is indifferent between q and q' at prices $p(t)$. Then $q'\in F(t)$, because $q\in F(t)$, so $G(y)\subseteq F(t)$. To see that $F(y)\subseteq F(t)$ take any $q''\in F(y)$. Then b strictly prefers q'' to q at prices y (recall that $q\notin F(y)$), so $v_{bq''} -p_{q''}(t) \geq v_{bq''} - y_{q''} > v_{bq} - y_q = v_{bq}-p_q(t)$, so b strictly prefers q'' to q at prices $p(t)$. (In the last inequality we used that $q\in U(t)$). Since $q\in F(t)$, it follows that $q''\in F(t)$. Therefore,

$F(y)\cup G(y)\subseteq F(t)$ and we have obtained the desired contradiction. Hence, $A_{b,i}(t)\subseteq G(t)$ and we have proved (2). Then,

$$\mathbf{C\subseteq G(t),} \qquad \textbf{(3)}$$

We also have that

C is contained in the set of elements of list $B_b(y)$, (4)

because otherwise all elements of C are out of $F(y)\cup G(y)$, so $v_{bq''} - y_{q''} > v_{bq} - y_q$, $\forall q''\in F(y)\cup G(y)$ and $\forall q\in C$. In this case we would have that $\forall q''\in F(y)\cup G(y)$ and $q\in C$ we can use that $p_q(t)=y_q$ and $y_{q''}\geq p_{q''}(t)$ to get that $v_{bq''} - p_{q''} \geq v_{bq''} - y_{q''} > v_{bq} - y_q = v_{bq} - p_q(t)$, so $q''\in F(t)$ (we used here that $q\in C\subseteq G(t)$). This implies that $F(y)\cup G(y)\subseteq F(t)$, so $|F(y)\cup G(y)|\leq|F(t)|<\mathrm{r(b)}$, absurd.

Now observe that if there is some $q\in C$ such that $q\in F(y)$, then there is some (b,j) such that $A^*_{b,j}(y)=\{q\}\subseteq C=A_{b,i}(t)\cap U(t)$, which contradicts (1). Hence, it follows by (4) that

$$\mathbf{C\subseteq G(y).} \qquad \textbf{(5)}$$

It also follows from (4) (or (5)) that all of C are demanded by b at prices y, so every element of C must be in some $A^*_{b,j}(y)$ for some (b,j). Let (b,j) be such that $A^*_{b,j}(y)\cap C\neq\phi$. Since $A^*_{b,j}(y)$ is not contained in C, by (1), we must have that $|A^*_{b,j}(y)|>1$, so $j=r(b)$. Then $C\subseteq A^*_{b,r(b)}(y)$ and we can write $A^*_{b,r(b)}(y)=C\cup D$, where $D\neq\phi$ and $D\cap C=\phi$. It is clear that $A^*_{b,r(b)}(y)\subseteq G(y)$, because $(b,r(b))$ is the last copy of b.

Set:

$\Gamma\equiv\{(b,j);\ A_{b,j}(t)\subseteq G(t)$ and $A_{b,j}(t)\cap U(t)\neq\phi\}$

$\Gamma'\equiv\{(b,j);\ A^*_{b,j}(y)\subseteq G(y)$ and $A^*_{b,j}(y)\cap U(t)\neq\phi\}$

$\Im\equiv\{(b,j);\ A_{b,j}(t)\subseteq G(t)$ and $A_{b,j}(t)\cap U(t)=\phi\}$

$\Im'\equiv\{(b,j);\ A^*_{b,j}(y)\subseteq G(y)$ and $A^*_{b,j}(y)\cap U(t)=\phi\}$

We have that $\Gamma\neq\phi$, since $(b,i)\in\Gamma$ by (2) and by the fact that $(b,i)\in T'$. Also, $\Gamma'\neq\phi$, since $(b,r(b))\in\Gamma'$. We are going to show that we can define a one-to-one map from $\Gamma-\{(b,r(b)\}$ into $\Gamma'-\{(b,r(b)\}$. In fact, it is clear that

$$\mathbf{|\Gamma|=r(b)-|F(t)|-|\Im| \text{ and } |\Gamma'|=r(b)-|F(y)|-|\Im'|.} \qquad \textbf{(6)}$$

We claim that

$$\mathbf{|F(t)|\geq|F(y)| + |\Im'| + |D-U(t)|} \qquad \textbf{(7)}$$

To see this, first observe that, due to the fact that $(b,r(b))\in\Gamma'$, then every $A^*_{b,j}(y)$ with (b,j) in $\Im'$ is singleton, so $|\bigcup_{(b,j)\in\Im'} A^*_{b,j}(y)|=|\Im'|$. Furthermore, $[\bigcup_{(b,j)\in\Im'} A^*_{b,j}(y)]\cap(D-U(t))=\phi$. Thus, since $G(y)-U(t)=[\bigcup_{(b,j)\in\Im'} A^*_{b,j}(y)]\cup(D-U(t))$, it follows that $|G(y)-U(t)|=|\Im'|+|D-U(t)|$. Therefore, it is enough to prove that $F(t)\supseteq F(y)\cup[G(y)-U(t)]$. That $F(t)\supseteq F(y)$ follows from (A.1) of Remark 5. Then, let $q\in G(y)-U(t)$. It also follows from (A.1) of Remark 5 that $q\in F(t)\cup G(t)$. If q was in $G(t)$ then, for all q' in C, $v_{bq'}-y_{q'}=v_{bq'}-p_{q'}(t)=v_{bq}-p_q(t)>v_{bq}-y_q=v_{bq'}-y_{q'}$, contradiction, where in the second equality we used (3), in the inequality we used that $q\notin U(t)$ and in the last equality we used that $q\in G(y)$ and (5). Therefore, $q\notin G(t)$, so $q\in F(t)$, and so $G(y)-U(t)\subseteq F(t)$. Hence, $F(t)\supseteq F(y)\cup[G(y)-U(t)]$ and we have proved (7).

Using (6) and (7), we get that $|\Gamma|=r(b)-|F(t)|-|\Im|\leq r(b)-|F(y)|-|\Im'|-|D-U(t)|-|\Im|=|\Gamma'|-|D-U(t)|-|\Im|$. That is,

$$|\Gamma|\leq|\Gamma'|-|D-U(t)|-|\Im|. \qquad (8)$$

Since $A_{b,r(b)}(t)\subseteq G(t)$ (because $(b,r(b))$ is the last copy of b), there must be that, either $(b,r(b))\in\Im$, or $(b,r(b))\in\Gamma$. If $(b,r(b))\in\Im$ then $|\Im|\geq1$ and $|\Gamma|=|G(t)\cap U(t)|$, because every $A_{b,j}(t)$ is singleton for $b\neq r(b)$. But then, $|\Gamma|\leq|\Gamma'|-|D-U(t)|-1\leq|\Gamma'|-1<|\Gamma'|$, so $|\Gamma'|>|\Gamma|$. Since $G(t)\cap U(t)=G(y)\cap U(t)$ by (A.2) of Remark 5, we must have that $|\Gamma'|>|\Gamma|=|G(t)\cap U(t)|=|G(y)\cap U(t)|$, which would be a contradiction (for each $(b,j)\in\Gamma'$, $A^*_{b,j}(y)$ has at least one element of $G(y)\cap U(t)$). Then, $(b,r(b))\in\Gamma$. In this case, $|\Gamma-\{(b,r(b))\}|\leq|\Gamma'-\{(b,r(b))\}|$. Hence, **we can define a one-to-one map f from $\Gamma-\{(b,r(b)\}$ into $\Gamma'-\{(b,r(b)\}$.** Then set:

$$A'_{b,j}(t)\equiv A_{b,j}(t) \text{ if } A_{b,j}(t)\subseteq F(t) \text{ or } (b,j)\in\Im.$$

$$A'_{b,j}(t)\equiv A^*_{f(b,j)}(y) \text{ if } (b,j)\in\Gamma-\{(b,r(b))\}.$$

$$A'_{b,r(b)}(t)\equiv G(t)-\bigcup_{j\neq r(b)} A'_{b,j}(t).$$

It is a matter of verification that $A'_b(t)$ is well defined (use that $G(t)\cap U(t)=G(y)\cap U(t)$ given by (A.2) of Remark 5). We have to check that, if $(b,j)\in T'$ and $j\neq r(b)$, there exists some (b,h) such that $A'_{b,j}(t)=A^*_{b,h}(y)\subseteq U(t)$; if $(b,r(b))\in T'$, there exists some (b,h) such that $A^*_{b,h}(y)\subseteq A'_{b,r(b)}(t)\cap U(t)$. Then, let $(b,j)\in T'$. This means that

$A_{b,j}(t)\cap U(t)\neq\phi$. If $A_{b,j}(t)\subseteq F(t)$, then $j\neq r(b)$, so $A_{b,j}(t)=\{q\}$ for some $q\in U(t)$, and so $q\in F(t)\cap U(t)= F(y)\cap U(t)$, where the equality follows from (A.2) of Remark 5. Thus, there is h such that $A^*_{b,h}(y)=\{q\}$ and so $A'_{b,j}(t)= A^*_{b,h}(y)\subseteq U(t)$, by definition of $A'(t)$. If $(b,j)\in T'$ and $A_{b,j}(t)\subseteq G(t)$, then $(b,j)\in\Gamma$ because $A_{b,j}(t)\cap U(t)\neq\phi$. We distinguish two cases:

Case 1. $j\neq r(b)$. Then, $A'_{b,j}(t)= A^*_{f(b,j)}(y)$ and $f(b,j)\in\Gamma'-\{(b,r(b))\}$, so $A^*_{f(b,j)}(y)=\{q\}$, for some q. The definition of Γ' implies that $q\in G(y)\cap U(t)$. Hence, $A'_{b,j}(t)= A^*_{f(b,j)}(y)\subseteq U(t)$.

Case 2. $j=r(b)$. The definition of A'_b implies that

$$\mathbf{A'_{b,r(b)}(t)\cap U(t) = [G(t)\cap U(t)] - \bigcup_{(b,k)\in\Gamma-(b,r(b))} A^*_{f(b,k)}(y)} \qquad \mathbf{(9)}$$

Suppose that $|D-U(t)|=0$. Then, we have that $A^*_{b,r(b)}(y)=C\cup D \subseteq U(t)$. Therefore, $A^*_{b,r(b)}(y)=[G(y)\cap U(t)]-\bigcup_{k\neq r(b)} A^*_{b,k}(y)=[G(y)\cap U(t)] - \bigcup_{(b,k)\in\Gamma'-(b,r(b))} A^*_{b,k}(y) \subseteq[G(y)\cap U(t)] - \bigcup_{(b,k)\in\Gamma-(b,r(b))} A^*_{f(b,k)}(y)$. Using (9) and the fact that $G(y)\cap U(t)=G(t)\cap U(t)$, given by (A.2) of Remark 5, we get that $A^*_{b,r(b)}(y) \subseteq A'_{b,r(b)}(t)\cap U(t)$.

Now, suppose that $|D-U(t)|>0$. Then $|\Gamma| \leq |\Gamma'| - 1$ by (8), so $|Im(f)| \leq|\Gamma'-\{(b,r(b))\}|-1$.[14] Thus, there is at least one $(b,h)\in\Gamma'$, with $h\neq r(b)$, such that $(b,h)\notin Im(f)$. The fact that $h\neq r(b)$ implies that $A^*_{b,h}(y)=\{q\}$, for some q. The fact that $(b,h)\notin Im(f)$ implies that $q\notin \bigcup_{(b,k)\in\Gamma-(b,r(b))} A^*_{f(b,k)}(y)$. That $(b,h)\in\Gamma'$ implies that $q\in G(y)\cap U(t)$ and so $q\in G(t)\cap U(t)$ by (A.2) of Remark 5. That is, $q\in[G(t)\cap U(t)] - \bigcup_{(b,k)\in\Gamma-(b,r(b))} A^*_{f(b,k)}(y)$. Now, use (9) to conclude that $A^*_{b,h}(y)=\{q\}\subseteq A'_{b,r(b)}(t)\cap U(t)$.

Thus, if $j=r(b)$, there is some (b,h) such that $A^*_{b,h}(y)\subseteq A'_{b,r(b)}(t)\cap U(t)$.

Hence we have demonstrated the desired result and the proof is complete. ■

[14] We are using the abbreviation $Im(f)$ to denote the image set of f: $f(\Gamma-\{(b,r(b))\})$.