Phase transition of demand explained by the heterogeneity of consumers' intrinsic preferences

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Abstract

In 1991 Gary S. Becker presented *A Note on Restaurant Pricing and Other Examples of Social Influences on Price* explaining why many successful restaurants, plays, sporting events, and other activities do not raise their prices even with persistent excess demand. The main reason for this is due to the discontinuity of stable demands, which is explained in Becker's (1991) analysis.

In the present paper we construct a discrete time stochastic model of socially interacting consumers deciding for one of two establishments. With this model we show that the discontinuity of stable demands, proposed by Gary S. Becker, depends crucially on an additional factor: the dispersion of the *consumers' intrinsic preferences* for the establishments.

Keywords: Heterogeneity of consumers' preferences, social interactions, social influence on price, discontinuity of demand, multiple demand equilibria.

JEL Classification codes: D11, D4, M30

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1 Introduction

As explained by Gary S. Becker (1991) in *A Note on Restaurant Pricing and Other Examples of Social Influences on Price*, the individual demands for establishments like restaurants, bars, night clubs, etc. are usually increasing functions of their aggregate demands. From this fact, Becker deduces that the resulting equilibrium demand which emerges in such cases, i.e., when the demand is positively auto-correlated, may appear to be discontinuous. This would explain why, for example, some restaurants do not raise price even when a given excess demand persists.

His explanation is essentially the following: if the demand is a continuous function of price, an over demanded restaurant could gradually raise prices in order to reduce its queue and increase profits without losing effective sales. On the other hand, supposing the existence of *social interaction among consumers.*, a slight increase in prices could reduce not only the queue, but also the number of consumers who visit the restaurant motivated by the fact that the restaurant is over demanded. The resulting effect is that a slight increase in prices may reduce significantly (discontinuously) the restaurant's demand, bringing it below the restaurant's capacity and also below the profit maximizing sales level. This would explain the puzzle proposed by Gary S. Becker.

We claim that in addition to the social interaction among consumers, there is another factor that is crucial in determining when and why the demand for such establishments appears to be discontinuous in price. Namely, we show that if the consumers are sufficiently heterogeneous in their *intrinsic preferences* for establishments then the demand function is no longer discontinuous. Contrary to this case, that is, if consumers are relatively homogeneous in their intrinsic preferences, then the aggregate demand appears to be a discontinuous function of price.

The *intrinsic preference* of a consumer is defined in our model as the difference in prices between two establishments (or two distinct consumption options) for which this consumer is indifferent to the choice between them, when both establishments have the same level of aggregate demand, i.e., when both establishments are equally attractive from a social point of view.

The relationship between Becker's (1991) work and ours is the following: Gary S. Becker motivated his analysis based on the demand/price behavior of two similar restaurants located closely to one another. He began his analysis investigating two intriguing facts: i) although the restaurants are similar in their services and amenities and charge similar prices, one of them is over demanded, while the other one is under demanded; and ii) the over demanded restaurant does not raise its prices. As already mentioned, his explanation for this phenomenon is that the social interaction among consumers may lead to a concentration of the resulting aggregate demands on one of the restaurants (observed in item i)) and to a discontinuous behavior of the aggregate demands of both restaurants in respect to their prices (this would explain why the over demanded restaurant does not raise prices, item ii)). Now, starting from our model, if we consider two *similar* (in an extreme case, identical) establishments competing for consumers in a market duopoly, then it is natural to suppose that all potential consumers of both establishments are relatively indifferent to the establishments, when both have the same level of aggregate demand. Therefore, when establishments are *similar*, it is also natural to suppose that the consumers are relatively homogeneous in their intrinsic preferences for establishments (all consumers have intrinsic preferences for the establishments close to zero). From this fact and also from the social interaction among consumers we would also deduce a concentration of the resulting aggregate demand on one of two restaurants, and that the aggregate demands of both restaurants are discontinuous in respect to their prices.

We propose now to extend this analysis to the case when the restaurants offer different services and amenities and/or when they are not located closely to one another, i.e., when they represent different consumption options for consumers, even when both establishments are equally demanded. Note that in this case, i.e., when the establishments are dissimilar, the consumers may have quite different intrinsic preferences for the establishments. This is illustrated by the example below.

Imagine a duopoly consisting of two competing night clubes in a rural area where there are no other night life options. Suppose also that these night clubes are similar in their services and amenities, but located relatively far way from each other. In this example, the intrinsic preference of a consumer is the difference in prices between the two night clubes that would compensate for the difference in the distance between each night club and the consumer's house, when both night clubs are equally demanded. If all the potential consumers of both night clubs live in the same small village (it does not matter where in this rural area), then the dispersion of the consumers' intrinsic preferences is relatively low. On the other hand, when their homes are uniformly distributed over this rural area, then the dispersion of the consumers' intrinsic preferences for the night clubs is relatively high. According to our model, in this case, the dispersion of the differences of the distances between each night club and each consumers's home will play a crucial role in determining when and why the aggregate demand appears to be concentrated on one of the night clubs and when and why the demands appear to be discontinuous in the night clubs' prices.

The model we are going to present concerns the time evolution of the market shares of two establishments competing for many consumers in a market duopoly. In this model, the values of the consumers' intrinsic preferences for the establishments are statistically distributed according to a probability distribution function. The model is constructed from the microscopic definitions of the consumers' intrinsic preferences and the consumers' discrete choices for one of the establishments. From the model's microscopic set-up and from the probability distribution of the consumers' intrinsic preferences, we deduce the macroscopic behavior of the aggregate demands for the establishments. The approach of our model is related to the work of Glaeser and Scheinkman (2000), and Brock and Durlauf (2000).

Section 2 contains the formal definition of the model. In Section 3, we derive the stochastic process of the difference of demand fractions which emerges in the model. In Section 4, we show that this stochastic process can be approximated by a specific dynamical system, when the number of consumers increases. In Section 5, we analyze the equilibria of this dynamical system. In Section 6, we deduce some socioeconomic results. In particular, we explain in Section 6 when and why the aggregate demand appears to be concentrated on one of the two establishments and when and why the aggregate demand appear to be discontinuous in respect to the establishments' prices.

2 The model

We model the situation where two establishments $E^{(+)}$ and $E^{(-)}$, say, two fashionable night clubs, are competing for consumers in a market duopoly.

Let us denote the set of all potential consumers of both establishments by $C := \{1, 2, \ldots, |C|\}$. We assume that at each time $t = 0, 1, 2, \ldots$, say, weekends, N consumers selected from C, $N \leq |C|$, decide for one of two establishments[2]. We shall denote by $C_t^{(N,C)}$ the set of consumers selected at time t. For example, if $|C| = 100, N = 3$ and the consumers 5, 20 and 99 are selected at time 1 then $C_1^{(3,C)} = \{5; 20; 99\}$. The selection rule may be quite general but must be independent of the consumers' intrinsic preferences. This independence will be formalized later in (8), after we have defined the intrinsic preferences of consumers.

Let us denote the number of consumers that choose establishment $E^{(+)}$ and $E^{(-)}$ at time t by $N_t(+1)$ and $N_t(-1)$, respectively ($N_t(+1)+N_t(-1) = N$, $\forall t$). Let us also assume that all potential consumers of both establishments (all elements of C) are aware of the popularity of both establishments in the recent past. In order to reflect this in our model, we assume that at each time $t > 0$, each consumer $i \in C$ knows the values of the fractions $N_{t-1}(+1)/N$ and $N_{t-1}(-1)/N$, where the values of $N_0(+1)/N$ and $N_0(-1)/N$, i.e., the initial demand fractions of the establishments $E^{(+1)}$ and $E^{(-1)}$, are two real numbers from the interval $[0, 1]$, totaling one, that are independent of N.

Let us now present the preference structure of the consumers. For $t \geq 1$ and $i \in C_t^{(N,C)}$, let us denote by $U_t^{(i)}(d)$ the utility of the consumer i in taking decision $d \in \{+1, -1\}$ at time t, i.e., in choosing the establishment $E^{(d)}$ at time t and queueing, if necessary (here and from now on, $E^{(+1)} := E^{(+)}$ and $E^{(-1)} := E^{(-)}$).

To focus our attention on the main phenomena we are going to explain, we propose the following utility functions:

$$\begin{gathered} U_t^{(i)}(d) := J\frac{N_{t-1}(d)}{N} - p(d) + u^{(i)}(d), \\ \forall\, t \in \{1, 2, \ldots\},\ \forall i \in C_t^{(N,C)},\ \forall\, d \in \{-1, +1\}, \end{gathered} \tag{1}$$

where J, $p(d)$ and $u_t^{(i)}$ are explained below.

J is a positive parameter, which measures the level of social influence on the utilities of consumers. This interpretation of J is clear from the fact that $N_{t-1}(d)/N$ expresses the popularity of the establishment $E^{(d)}$ ($d \in \{+1, -1\}$, $t \geq 0$) in the immediate past. The assumption of the positivity of J follows Becker's (1991) explanation that assumes that "...a consumer's demand for some good depends (positively) on the demands by other consumers. The motivation for this approach is the recognition that restaurant eating, watching a game or play, attending a concert, or talking about books are all social activities in which people consume a product or service together and partly in public."

[2]We suppose that the fraction of the potential consumers that indeed decide to consume at time t is more or less the same for all t's. Accordingly, N does not depend on time in our model.

$p(d)$ is positive and denotes the price charged by the establishment $E^{(d)}$ ($d \in \{+1, -1\}$).

$u^{(i)}(d) \in \mathbb{R}$ is a consumer's i specific utility increment in choosing establishment $E^{(d)}$ at time t ($i \in C_t^{(N,C)}$, $d \in \{+1, -1\}$, $t \geq 0$).

For $t \geq 1$ and $i \in C_t^{(N,C)}$, let us denote by $d_t^{(i)} \in \{+1, -1\}$ the decision of consumer i at time t; that is, if $d_t^{(i)} = +1$ (-1) then the consumer i goes to establishment $E^{(+)}$ ($E^{(-)}$ respectively) (and queues if necessary). The utility maximization behavior implies that

$$\begin{array}{lcl} U_t^{(i)}(+1) - U_t^{(i)}(-1) > 0 & \Rightarrow & d_t^{(i)} = +1 \\ U_t^{(i)}(+1) - U_t^{(i)}(-1) < 0 & \Rightarrow & d_t^{(i)} = -1 \end{array} \tag{2}$$

To proceed, we need to define the consumer's decision when $U_t^{(i)}(+1) - U_t^{(i)}(-1) = 0$. We assume that

$$U_t^{(i)}(+1) - U_t^{(i)}(-1) = 0 \quad \Rightarrow \quad d_t^{(i)} = -1 \tag{3}$$

As we will see, the assumption (3) does not cause an asymmetry in the resulting aggregate demand, since in accordance with the further descriptions of the model, the event $\{U_t^{(i)}(+1) - U_t^{(i)}(-1) = 0 \text{ for some } i \in C_t^{(N,C)}\}$ occurs with probability zero.

Now, from the utility functions defined in (1) and the relationship (2) and (3) it follows that, for $t \geq 1$ and $i \in C_t^{(N,C)}$, it holds that

$$d_t^{(i)} = \begin{cases} +1, & \text{if} \quad J\frac{N_{t-1}(+1) - N_{t-1}(-1)}{N} - \theta + \theta^{(i)} > 0 \\ -1, & \text{otherwise} \end{cases} \tag{4}$$

where $\theta := p(+1) - p(-1)$ and $\theta^{(i)} := u^{(i)}(+1) - u^{(i)}(-1)$.

Remark 1. *Note that $\theta^{(i)}$ determines the preference of consumer i over the establishment/price pairs $(E^{(d)}, p(d))$, $d \in \{-1, 1\}$, when $\frac{N_{t-1}(+1)}{N} = \frac{N_{t-1}(-1)}{N}$, that is, when the consumer is free of social influence. In fact we have:*

$$\frac{N_{t-1}(+1)}{N} = \frac{N_{t-1}(-1)}{N} \quad \Rightarrow \quad (E^{(-1)}, p(-1)) \begin{cases} \succ_i (E^{(+1)}, p(+1)), & \text{if } \theta > \theta^{(i)} \\ \sim_i (E^{(+1)}, p(+1)), & \text{if } \theta = \theta^{(i)} \\ \prec_i (E^{(+1)}, p(+1)), & \text{if } \theta < \theta^{(i)} \end{cases} \tag{5}$$

where $(E^{(-1)}, p(-1)) \succ_i (E^{(+1)}, p(+1))$ means that consumer i prefers to go to $E^{(-1)}$ at the price $p(-1)$ than to $E^{(+1)}$ at the price $p(+1)$. With $\prec_i$ we denote the reverse preference relationship for consumer i, and with $\sim_i$ we denote the indifference of consumer i between the two establishment/price pairs.

In accordance with the above,

$\theta^{(i)}$ will be called the *intrinsic preference* of consumer i ($i \in C$)

Note that $\theta^{(i)}$ is time independent and is defined for each potential consumer $i \in C$. We anticipate that the degree of dispersion of the consumers' intrinsic preferences, $\theta^{(i)}$, $i \in C$, will play a crucial role in determining if the demands for the establishments are continuous functions of their prices or not.

In order to model the heterogeneity of the consumers' intrinsic preferences, we assume that $\theta^{(i)}$, $i \in C$, are independent and identically distributed random variables, each composed of a *mean intrinsic preference* $\overline{\theta}$ and a consumer's i specific deviation $\xi^{(i)}$ from this mean ($i \in C$), i.e.,

$$\theta^{(i)} = \overline{\theta} + \xi^{(i)}, \quad i \in C \tag{6}$$

where $\xi^{(i)}$, $i \in C$, are independent and identically distributed random variables. We assume that their distribution function Φ is differentiable and that its derivative Φ' satisfies the following properties:

$$\begin{array}{rll} \text{1) Unimodality:} & \Phi' \quad \text{is increasing in } (-\infty, 0] \text{ and decreasing in} \quad [0, \infty) \\ \text{2) Symmetry:} & \Phi'(x) = \Phi'(|x|) \quad \forall x \in \mathbb{R} \end{array} \tag{7}$$

The symmetry of Φ' implies that $\overline{\theta} = \mathbb{E}(\theta^{(i)})$, $i \in C$, where $\mathbb{E}(\theta^{(i)})$ denotes the expected value of the random variable $\theta^{(i)}$. This fact justifies the name "mean intrinsic preference" given to $\overline{\theta}$. In our model, $\overline{\theta}$ is a time independent parameter. It allows us to model exogenous intervention like promotions, advertising, etc., that may shift the mean intrinsic preference of consumers without affecting the deviations $\xi^{(i)}, i \in C$, from this mean.

With respect to the assumptions (7), we note the following. We expect that a unimodal distribution would be a good first order approximation of a real distribution. The symmetry is assumed in order to simplify our exposition. Analogous results to those that we shall present can be derived assuming only unimodality of Φ.

Once we have defined the intrinsic preferences of consumers, we can now formalize the precondition for the selection of the subset $C_t^{(N,C)}$ of N consumers from the population C. Recall that this precondition was imposed at the end of the second paragraph of the present section. It postulated that the selection of $C_t^{(N,C)}$ may be deterministic or random, but must be independent of the intrinsic preferences $\theta^{(i)}, i \in C$. Since $\theta^{(i)} = \bar{\theta} + \xi^{(i)}$ and since $\bar{\theta}$ is a constant, this independence can be formalized as follows: for any time t and any subset C' of N elements of $C = \{1, 2, \ldots, |C|\}$ ($N \leq |C|$), it holds that

$$\mathbb{P}\left(C_t^{(N,C)} = C' \,\middle|\, \xi^{(1)}, \ldots, \xi^{|C|}\right) = \mathbb{P}\left(C_t^{(N,C)} = C'\right) \tag{8}$$

The equality (8) means that the probability of selecting a given subset C' from the set of all potential consumers C at time t does not depend on the intrinsic preferences of consumers, $\theta^{(i)}$, $i \in C$.

3 The stochastic process of the market share

In this section, we shall present a description of the stochastic process of the market share of the establishments over time, i.e., the stochastic process of $N_t(+1)/N$ and $N_t(-1))/N$. Recall that $N_t(+1)$ and $N_t(-1)$ denote the number of consumers that choose establishment $E^{(+)}$ and $E^{(-)}$ at time t, respectively, and that $N = N_t(+1) + N_t(-1)$.

In order to allow us an easy treatment of the stochastic process of the demand fractions $N_t(+1)/N$ and $N_t(-1))/N$, we will describe the equivalent stochastic process of *the difference of demand fractions:*

$$m_t^{(N,C)} = (N_t(+1) - N_t(-1))/N \tag{9}$$

That the stochastic process of $m_t^{(N,C)}$ is equivalent to the stochastic processes of $N_t(+1)/N$ and $N_t(-1)/N$ is due to the fact that: $N_t(+1)/N + N_t(-1))/N = 1$, $\forall t \geq 0$. In fact, the latter relation implies that the value of $m_t^{(N,C)}$ follows from the values of $N_t(+1)/N$ and $N_t(-1)/N$ and vice versa, $\forall t \geq 0$.

From the construction of our model, the difference of demand fractions is equal to the arithmetic mean of decisions:

$$m_t^{(N,C)} = \sum_{i \in C_t^{(N,C)}} d_t^{(i)}/N \tag{10}$$

The relationships (4), (6) and (10) altogether allow us to present this process in the following way. First of all generate independently $|C|$ random variables $\xi^{(1)}, \xi^{(2)}, \ldots \xi^{(|C|)}$ with distribution function Φ; for $t = 0$, set $m_0^{(N,C)} = m_0$, where here and from now on, $m_0 := (N_0(+1) - N_0(-1))/N \in [-1, 1]$; for $t > 0$, determine $m_t^{(N,C)}$ by the following steps:

1) Choose the set $C_t^{(N,C)}$, of N consumers from the population $C = \{1, 2, \ldots, |C|\}$ in accordance with the specified selection rule.

2) Use the values of $m_{t-1}^{(N,C)}$ and $\xi^{(i)}, i \in C_t^{(N,C)}$, to define

$$d_t^{(i)} := \begin{cases} +1, & \text{if} \quad -\xi^{(i)} < J m_{t-1}^{(N,C^N)} - h \\ -1, & \text{otherwise} \end{cases}, \qquad \forall i \in C_t^{(N,C)}$$

where $h := \theta - \bar{\theta} = p(+1) - p(-1) - \bar{\theta}$.

3) Set $m_t^{(N,C)} := \dfrac{\sum_{i \in C_t^{(N,C)}} d_t^{(i)}}{N}$

The description 1)-3) expresses clearly the structure of the randomness of the process $(m_t^{(N,C)})_{t\geq 0}$. We would like now to make certain remarks in respect of this.

To start with, we state explicitly that the random variables $\xi^{(i)}$, $i \in C := \{1, 2, \ldots |C|\}$ do not change their values over time. This is natural, since $\xi^{(i)}$ determines the intrinsic preference of the consumer i. Thus, for each t, the randomness of the transition from $m_{t-1}^{(N,C)}$ to $m_t^{(N,C)}$ stems from the selection rule at time t (that can be random). This fact makes the discussion of three particular cases that we present below important:

When $N = |C|$ $(C_t^{(N,C)} = C$**,** $t \geq 0)$,

then there is no randomness over time; i.e., all $m_t^{(N,C)}$, $t = 1, 2, \ldots$ are uniquely determined by the initial value of $m_0^{(N,C)} = m_0$ and the values of J, h and $\xi^{(1)}, \xi^{(2)}, \ldots \xi^{(|C|)}$.

When $|C| = \infty$ **and the selection rule is such that** $C_s^{(N,C)} \cap C_t^{(N,C)} = \emptyset$ **for** $s \neq t$,

then, for any time t, the consumers visiting the establishments have never visited them before. In this case, the process $(m_t^{(N,C)})_{t\geq 0}$ has a stochastic time update and is Markovian.

When $0 < N/|C| < 1$,

then the establishments may be visited by old and new consumers.

Of the three cases, this case approximates most to real life.

With respect to the existence of these three different cases of the process $(m_t^{(N,C)})_{t\geq 0}$, we note that the property that will be essential for our study of the process holds for all three of them. This property is the essence of Proposition 1 stated in the next section.

4 The dynamics of the difference of demand fractions in large populations

Proposition 1, presented below, shows that the dynamical system defined by $m_t = 2\Phi(Jm_{t-1} - h) - 1$, $\forall t > 0$, approximates the stochastic process $(m_t^{(N,C)})_{t\geq 0}$ for a finite but large enough number of consumers N ($N < |C|$).

Proposition 1. *Let $\xi^{(i)}$, $i = 1, 2, \ldots$ be independent and identically distributed random variables with the common cumulative probability distribution function Φ that satisfies (7). Let $J \geq 0$ and $h \in \mathbb{R}$ be arbitrarily fixed. Let $\{|C^N|, N \in \mathbb{N}\}$ be an arbitrary sequence of natural numbers satisfying $N \leq |C^N|$ $\forall N$. For each $t \in \mathbb{N}$ and each $N \in \mathbb{N}$, let*

$$C_t^{N,C^N} = \{c_1^{t,N},\, c_2^{t,N}, \ldots, c_N^{t,N}\} \tag{11}$$

be an arbitrary set (deterministic or random) of N distinct numbers from $C^N = \{1, 2, \ldots, |C^N|\}$ which is independent from the ξ's. Finally, let $(m_t^{N,C^N})_{t\geq 0}$ be defined by the steps (2)-(3) of Section 3 with an initial value m_0.

Then,

$$\forall t \geq 0: \quad \lim_{N\to\infty} \left|\, m_t^{(N,C^N)} - g_t(m_0; h) \,\right| = 0 \quad \textit{almost surely}\ , \tag{12}$$

where $g_t(m_0; h)$ denotes the $t-$th iteration of the mapping $m \mapsto 2\Phi(Jm - h) - 1$, starting from m_0.

We give below the formal definition of $g_t(m_0; h)$ used in (12):

for $t \geq 1$, $g_t : [-1, 1] \times \mathbb{R} \to [-1, 1]$ is defined by

$$g_t(m_0; h) := g(g_{t-1}(m_0; h); h) \tag{13}$$

with $g_0(m_0; h) := m_0$ and $g : [-1, 1] \times \mathbb{R} \to [-1, 1]$ is defined by

$$g(m; h) := 2\Phi(Jm - h) - 1 \tag{14}$$

Proof of Proposition 1. First, we shall prove that for all $t = 1, 2, \ldots$ it holds that

$$\lim_{N\to\infty} \left|\, m_t^{(N,C^N)} - g(m_{t-1}^{(N,C^N)}; h) \,\right| = 0, \quad \text{almost surely} \tag{15}$$

The assertion (12) will then follow from (15) by induction on t.
Proof of (15). Let us fix an arbitrary value for t through out the proof of (15).

Let us choose an arbitrary value for N and consider

$$-\xi^{(c_1^{t,N})},\, -\xi^{(c_2^{t,N})}, \ldots, -\xi^{(c_N^{t,N})} \tag{16}$$

that is, consider the subset of random variables from $\{-\xi^{(1)}, -\xi^{(2)}, \ldots\}$ whose indexes are $c_1^{t,N}, c_2^{t,N}, \ldots, c_N^{t,N}$. Since our proposition assumes that the indexes c's are independent of the random variables ξ's, it follows that the variables in (16) are independent, and from the symmetry of Φ (the distribution function of ξ's) it follows that Φ is the distribution function of each one of the variables in (16). Thus, if for an arbitrary $x \in \mathbb{R}$ we define

$$D_t^{(i,N)}(x) := \begin{cases} 1, & \text{if} \qquad -\xi^{(c_i^{t,N})} < x \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

and

$$\Phi_N(x) := \frac{1}{N} \sum_{i=1}^{N} D_t^{(i,N)}(x) \tag{18}$$

then the independence of $D_t^{(1,N)}, D_t^{(2,N)}, \ldots, D_t^{(N,N)}$ and their boundness, i.e., $|D_t^{(i,N)}| \leq 1$, together with the Borel-Cantelli Lemma imply that

$$\forall x \qquad \Phi_N(x) \to \Phi(x) \text{ almost surely as } N \to \infty \tag{19}$$

Applying to (19) the argument that proves the Glivenko-Cantelli theorem, see Durrett (1995), we deduce that

$$\begin{aligned} &\lim_{N\to\infty} \sup_x \Big| \tfrac{1}{N} \textstyle\sum_{i=1}^{N} D_t^{(i,N)}(x) - \Phi(x) \Big| \\ &= \lim_{N\to\infty} \sup_x \Big| \Phi_N(x) - \Phi(x) \Big| = 0, \quad \text{almost surely} \end{aligned} \tag{20}$$

The uniform convergence (20) will be used below to prove (15).

We now observe that in accordance with the steps 1)-3) of the construction of the process $(m_t^{(N,C^N)})_{t\geq 0}$ described in Section 3, we can use the notations introduced in (17) to write

$$m_t^{(N,C^N)} = 2\Big[\frac{1}{N} \sum_{i=1}^{N} D_t^{(i,N)}(J m_{t-1}^{(N,C^N)} - h) \Big] - 1 \tag{21}$$

This expression and the definition of the mapping $m \mapsto g(m;h) = 2\Phi(Jm - h) - 1$ imply that

$$\begin{aligned} \big| m_t^{(N,C^N)} - g(m_{t-1}^{(N,C^N)}; h) \big| &= 2 \big| \tfrac{1}{N} \textstyle\sum_{i=1}^{N} D_t^{(i,N)}(m_{t-1}^{(N,C^N)} - h) - \Phi(m_{t-1}^{(N,C^N)} - h) \big| \\ &\leq 2 \sup_x \Big| \Phi_N(x) - \Phi(x) \Big| \end{aligned} \tag{22}$$

The inequality (22) and the uniform convergence in (20) imply the almost sure convergence (15).

The proof of (12) will be made by induction on t.

The induction basis ($t = 0$). For $t = 0$, the result is immediate: $m_0^{(N,C^N)} = m_0$ is an assumption of the proposition, while $g_0(m_0; h) = m_0$ follows from the definition of g_0 (immediately after (13)).

The induction step ($t-1 \curvearrowright t$). Suppose now that (12) holds for an arbitrary $t-1 \geq 0$, i.e., that $m_{t-1}^{(N,C^N)} - g_{t-1}(m_0; h)$ converges almost surely to zero (when $N \to \infty$). Since $m \mapsto g(m; h)$ is continuous then $g(m_{t-1}^{(N,C^N)}; h) - g_t(m_0; h) = g(m_{t-1}^{(N,C^N)}; h) - g(g_{t-1}(m_0; h); h)$ converges almost surely to zero, too.

Now, since according to (15), $g(m_{t-1}^{(N,C^N)}; h) - m_t^{(N,C^N)}$ converges almost surely to zero, we can apply the triangular inequality and deduce the result for t:

$$\begin{aligned} &|\, m_t^{(N,C^N)} - g_t(m_0; h) \,| \leq \\ &\qquad \leq |\, m_t^{(N,C^N)} - g(m_{t-1}^{(N,C^N)}; h) \,| + |\, g(m_{t-1}^{(N,C^N)}; h) - g_t(m_0; h) \,| \to 0 \end{aligned} \tag{23}$$

This completes the proof of Proposition 1. □

Proposition 1 states that the trajectories of the stochastic process $(m_t^{(N,C^N)})_{t\geq 0}$ converge pointwise in $t = 0, 1, 2 \ldots$, almost surely to the deterministic dynamical system $(m_t)_{t\geq 0}$ defined by

$$m_t = 2\Phi(J m_{t-1} - h) - 1, \quad \forall t > 0 \tag{24}$$

In what follows we will call the dynamical system $(m_t)_{t\geq 0}$ defined in (24) the *large population limit* of the stochastic process $(m_t^{(N,C)})_{t\geq 0}$, defined in Section 3.

5 Equilibria of demand and their stability in large populations

We proceed below by studying the *large population limit* of the stochastic process $(m_t^{(N,C)})_{t\geq 0}$, i.e., the dynamical behavior of the difference of demand fractions of establishments $E^{(+)}$ and $E^{(-)}$ for a large population of consumers. This study will be made by investigating the properties of the dynamical system $(m_t)_{t\geq 0}$, defined by the recursion (24) ($m_t = 2\Phi(Jm_{t-1} - h) - 1, \quad t = 1, 2, \ldots$).

We will say that $m \in [-1, +1]$ is a *large population equilibrium of difference of demand fractions*, if it is an equilibrium of dynamical system (24), that is, if

$$m = 2\Phi(Jm + h) - 1$$

Let us define the *domain of attraction* of an equilibrium $m \in [-1, +1]$ ($m = 2\Phi(Jm + h) - 1$) of (24) by the set

$$\{\, m_0 \in [-1, +1] \,|\, \lim_{t\to\infty} g_t(m_0; h) = m \,\},$$

where g_t, $t \geq 0$, are defined in (13), i.e., $g_t(m_0; h)$ is the t-the iteraction of $m \mapsto 2\Phi(Jm - h) - 1$, starting from $m = m_0$.

We will distinguish between globally stable, locally stable and unstable equilibria as follows. We will say that $m \in [-1, +1]$ is: 1) a *globally stable* equilibrium of (24) if, and only if the domain of attraction of m is the whole interval $[-1, +1]$; 2) a *locally stable* equilibrium of (24) if, and only if the domain of attraction of m contains a set of type $(m - \varepsilon, m + \varepsilon) \cap [-1, 1]$ for some $\varepsilon > 0$; and 3) m is an *unstable equilibrium* of (24) if, only if it is neither a globally nor a locally stable equilibrium of (24). We will also call a locally stable equilibrium of (24) an *attractor* of (24).

The next proposition, illustrated in Figures 1, 2 and 3, determines all possible globally stable, locally stable and unstable equilibria of difference of demand fractions of (24) depending on values assumed by $\Phi'(0)^{-1}$, J and h.

Note that, for a large class of parametric distribution functions, $\Phi'(0)^{-1}$ is an increasing function of the standard deviation of the consumers' intrinsic preferences, $\theta^{(i)}$, $i \in C$, whose distribution function is $x \mapsto \Phi(x - \overline{\theta})$. In particular, if Φ is the cumulative normal distribution with mean zero and variance σ^2, then $\Phi'(0)^{-1}$ is proportional to the standard deviation σ of the consumers' intrinsic preferences $\theta^{(i)}$, $i \in C$, ($\Phi'(0)^{-1} = \sqrt{2\pi}\sigma$). Supposing such a suitable parametric form for Φ, we can interpret $\Phi'(0)^{-1}$ as a measure for the heterogeneity in consumers' intrinsic preferences for the establishments.

Our interpretation of $(\Phi'(0))^{-1}$ allows us to establish the relationships between the consumer interactions (J), the heterogeneity of consumers' intrinsic preferences ($\Phi'(0)^{-1}$) and the resulting types of equilibria for the difference of demand fractions. In what follows, we will say, therefore, that

$\Phi'(0)^{-1}$ is *the heterogeneity of consumers' intrinsic preferences.*

Let us state now

Proposition 2. *Let Φ be a probability distribution function satisfying (7). Set $g(m;h) := 2\Phi\big(Jm - h\big) - 1$ and denote by $\mathcal{M} := \{\, m \,|\, m = g(m;h) \,\}$ the set of equilibria of the dynamical system (24) ($m_t = g(m_{t-1};h)$, $t \geq 1$). Denote by $|\mathcal{M}|$ the number of elements of $\mathcal{M}$. The following relationships hold for $J \in [0,\infty)$ and $h \in \mathbb{R}$:*

1. *If $\Phi'(0)^{-1} \geq 2J$*

   then $|\mathcal{M}| = 1$ and the unique element of $\mathcal{M}$, denoted by $\bar{M}(h)$, is a globally stable equilibrium of (24). Furthermore, the function $h \mapsto \bar{M}(h)$ is decreasing, odd and continuous in h (with $\bar{M}(0) = 0$). See Figure 1.a and 2 for this item.

2. *If $\Phi'(0)^{-1} < 2J$*

   then there are critical thresholds $h_* > 0$ and $M_* > 0$ determined by

$$\begin{cases} M_* &= 2\Phi\big(JM_* - h_*\big) - 1 \\ 1 &= 2\Phi'\big(JM_* - h_*\big)J \end{cases}$$

   such that

   (a) *if $-\infty < h < -h_*$*
   then $|\mathcal{M}| = 1$, and the unique element of $\mathcal{M}$, denoted by $M_+(h)$, is a globally stable equilibrium of (24). See figure 1.b for this item;

   (b) *if $h = -h_*$*
   then $|\mathcal{M}| = 2$, and the elements of $\mathcal{M}$, denoted by $M_-(-h)$ and $M_+(-h)$ are such that: (i) $M_-(-h) = -M_* < 0 < M_* < M_+(-h)$; (ii) $M_-(-h)$ ($= -M_*$) is an unstable equilibrium of (24) and its domain of attraction is $[-1, -M_*]$; (iii) $M_+(-h)$ is a locally stable equilibrium of (24) and its domain of attraction is $(-M_*, 1]$. See figure 1.c for this item;

   (c) *if $-h_* < h < h_*$*
   then $|\mathcal{M}| = 3$, and the elements of $\mathcal{M}$, denoted by $M_-(h)$, $M(h)$ and $M_+(h)$, are such that: (i) $M_-(h) < -M_* < M(h) < M_* < M_+(h)$, and (ii) $M(h)$ is unstable, while $M_-(h)$ and $M_+(h)$ are locally stable equilibria of (24); the domains of attraction of $M_-(h)$, $M(h)$ and $M_+(h)$ are $[-1, M(h))$, $\{M(h)\}$ and $(M(h), 1]$ respectively. See figure 1.d for this item;

   (d) *if $h = h_*$*
   then $|\mathcal{M}| = 2$, and the elements of $\mathcal{M}$, denoted by $M_-(h)$ and $M_+(h)$ are such that: (i) $M_-(h) < -M_* < 0 < M_* = M_+(h)$; (ii) $M_-(h)$is a locally stable equilibrium of (24) and its domain of attraction is $[-1, M_*]$; (iii) $M_+(h)$ ($= M_*$) is an unstable equilibrium of (24) and its domain of attraction is $(M_*, 1]$. See figure 1.e for this item;

   (e) *if $h_* < h < \infty$*
   then $|\mathcal{M}| = 1$, and the unique element of $\mathcal{M}$, denoted by $M_+(h)$, is a globally stable equilibrium of (24). See figure 1.f for this item;

(f) $M_+(h) = -M_-(-h)$ *for all* $h \in (-\infty, h_*]$*, and* $M(h) = -M(-h)$ *for all* $h \in (-h_*, +h_*)$*. Furthermore, the functions* $M_+ : (-\infty, h_*] \to [-1,1]$ *and* $M : (-h_*, +h_*) \to [-1,1]$ *satisfy the following properties:*

i. $M_+ : (-\infty, h_*] \to [-1,1]$ *is decreasing and continuous,*

ii. $M : (-h_*, +h_*) \to [-1,1]$ *is increasing and continuous,*

iii. $\lim_{h\to-\infty} M_+(h) = 1$ *and* $M_+(h_*) = M_* = \lim_{h\to h_*} M(h)$

See Figure 3 for this item.

Proof. The formal proof of Proposition 2 uses the the Implicit Function Theorem. What we present below is an informal but intuitive argument that support the assertion of the proposition.

The intuitive proof of Proposition 2 we are going to present uses the following three obvious facts. Firstly, m is an equilibrium of the dynamical system $m_t = 2\Phi(Jm_{t-1}-h)-1$ if and only if (m,m) is the interception point of the graph of the mapping $m \mapsto g(m;h) := 2\Phi(Jm-h)-1$ with the 45°-line $\{(m,m) : m \in [-1,1]\}$. Secondly, the graph of $m \mapsto g(m;h) := 2\Phi(Jm-h)-1$ may be obtained from the graph of $m \mapsto g(m;0) := 2\Phi(Jm)-1$ by a horizontal shift by h/J. Thirdly, the graph of $m \mapsto g(m;h)$ has an "S" shape, by which we mean the following three properties of the function $m \mapsto g(m;h)$: i) it is increasing and continuous; ii) it is bounded below by -1 and bounded above by $+1$; and iii) it is concave on $(-\infty, h/J]$ and it is convex on $[h/J, \infty)$.

We will divide the proof into the two cases:

$$1)\ \Phi'(0)^{-1} \geq 2J \quad \text{and} \quad 2)\ \Phi'(0)^{-1} < 2J.$$

$\Phi'(0)^{-1} \geq 2J$ (Figures 1.a and 2 help to explain our arguments in this case):

When $\Phi'(0)^{-1} \geq 2J$, then the function $m \mapsto \partial g(m,0)/\partial m$ attains its maximum at $m=0$, and $\partial g(m,0)/\partial m|_{m=0} = 2J\Phi'(0) \leq 1$. Thus, $(0,0)$ is the unique interception point between the graph of $m \mapsto g(m;0)$ (the dashed line in Figure 1.a) and the 45°-line. Since the graph of $m \mapsto g(m;h)$, can be obtained by horizontal shifting the graph of $m \mapsto g(m,0)$, it is easy to verify that for any $h \in \mathbb{R}$, there is only one interception between the graph of $m \mapsto g(m;h)$ and the 45°-line. Denoting this interception by $(\bar{M}(h), \bar{M}(h))$ and shifting h from the left to the right, it is easy to observe graphically that $h \mapsto \bar{M}(h)$ satisfies the properties stated in item (1) of Proposition 2. See the graph of $h \mapsto \bar{M}(h)$ in Figure 2. That the domain of attraction of $\bar{M}(h)$ is the whole intervale $[-1,1]$ can be deduced from a graphical analysis as indicated with the zig-zag lines in Figure 1.a.

$\Phi'(0)^{-1} < 2J$ (Figures 1.b-f and 3 help to explain our arguments in this case):

Contrary to the previous case, when $\Phi'(0)^{-1} < 2J$, we have here that $\partial g(m,0)/\partial m|_{m=0} = 2J\Phi'(0) > 1$. From the "S" shape of the graph of $m \mapsto g(m,0)$ it follows that there are three interceptions between the graph of $m \mapsto g(m,0)$ and the 45°-line. From the continuity Φ and the symmetry of Φ', it then follows that there are also three interceptions, $(M_-(h), M_-(h))$, $(M(h), M(h))$ and $(M_+(h), M_+(h))$, with $M_-(h) < M(h) < M_+(h)$, between the graph $m \mapsto g(m, h)$ and the 45°-line if h does not lie far from zero, i.e., if $|h| < h_*$. If $|h| = h_*$ there are two interceptions, $(M_-(h), M_-(h))$ and $(M_+(h), M_+(h))$, with $M_-(h) < M_+(h)$. If $|h| > h_*$, there is only one interception between the graph $m \mapsto g(m, h)$ and the 45°-line. If $h < -h_*$, we denote this interception by $(M_+(h), M_+(h))$; and, if $h > h_*$, we denote it by $(M_+(h), M_+(h))$.

The assertions concerning the location of the equilibria $M_-(h)$, $M(h)$ and $M_+(h)$ can be deduced from the graphs of $h \mapsto M_-(h)$, $h \mapsto M(h)$ and $h \mapsto M_+(h)$ illustrated in Figure 3. These graphs can be constructed by shifting the graph of $m \mapsto g(m, h)$ from the left to the right and observing the movement of the equilibrium in interest as illustrated in Figures 1.b-f. The domains of attraction of the equilibria $\bar{M}(h), M_-(h), M(h)$ and $M_+(h)$ are identified from the "S" shape of the graph of $m \mapsto g(m; h)$ applying a graphical analysis as indicated with the zig-zag lines in Figures 1.b and 1.f. □

6 Socioeconomic interpretation and applications

The contents of Proposition 1 and 2 allows us to view m_t as an approximation for $m_t^{(N,C)}$, when the population is large. Indeed, these propositions ensure that the distribution of the random variable $m_t^{(N,C)}$ is concentrated around m_t and that the probability of the deviations of $m_t^{(N,C)}$ from m_t may be made as small as desired by the choice of an appropriate number of consumers N. This implies that $\lim_{t\to\infty} m_t$ may be viewed as the time-stable difference of demand fractions of the establishments; to be precise: if the difference of demand fractions starts from some $m_0^{(N,C)} = m_0$ and if we wait for a given relaxation time until we see that $m_t^{(N,C)}$ is more or less constant in time, then the value of $m_t^{(N,C)}$ will be close to the $\lim_{t\to\infty} g_t(m_0; h)$. Since this limit is the subject of Proposition 2, its results provide us with properties of the stable distribution of the consumers between the establishments and its dependence on the model parameters. These properties will be analyzed below from the socioeconomic point of view.

6.1 Interpretation of the model's parameters

The results of Propositions 1 and 2 which we are going to interpret from the socioeconomic point of view involve the quantities J, $\Phi'(0)^{-1}$ and h. The socioeconomic interpretations of J and $\Phi'(0)$ have been already presented; recall that J gives the strength of social influence among consumers, while $\Phi'(0)^{-1}$ expresses the heterogeneity of the consumer's intrinsic preferences $\theta^{(i)}$, $i \in C$. Here, we shall interpret the parameter h from a socioeconomic point of view.

Recall we have defined $\theta := p(+1) - p(-1)$, $\bar{\theta} := \mathbb{E}(\theta^{(i)})$ and $h := \theta - \bar{\theta}$. Thus,

$$h = \mathbb{E}\big[(p(+1) - p(-1)) - \theta^{(i)}\big]$$

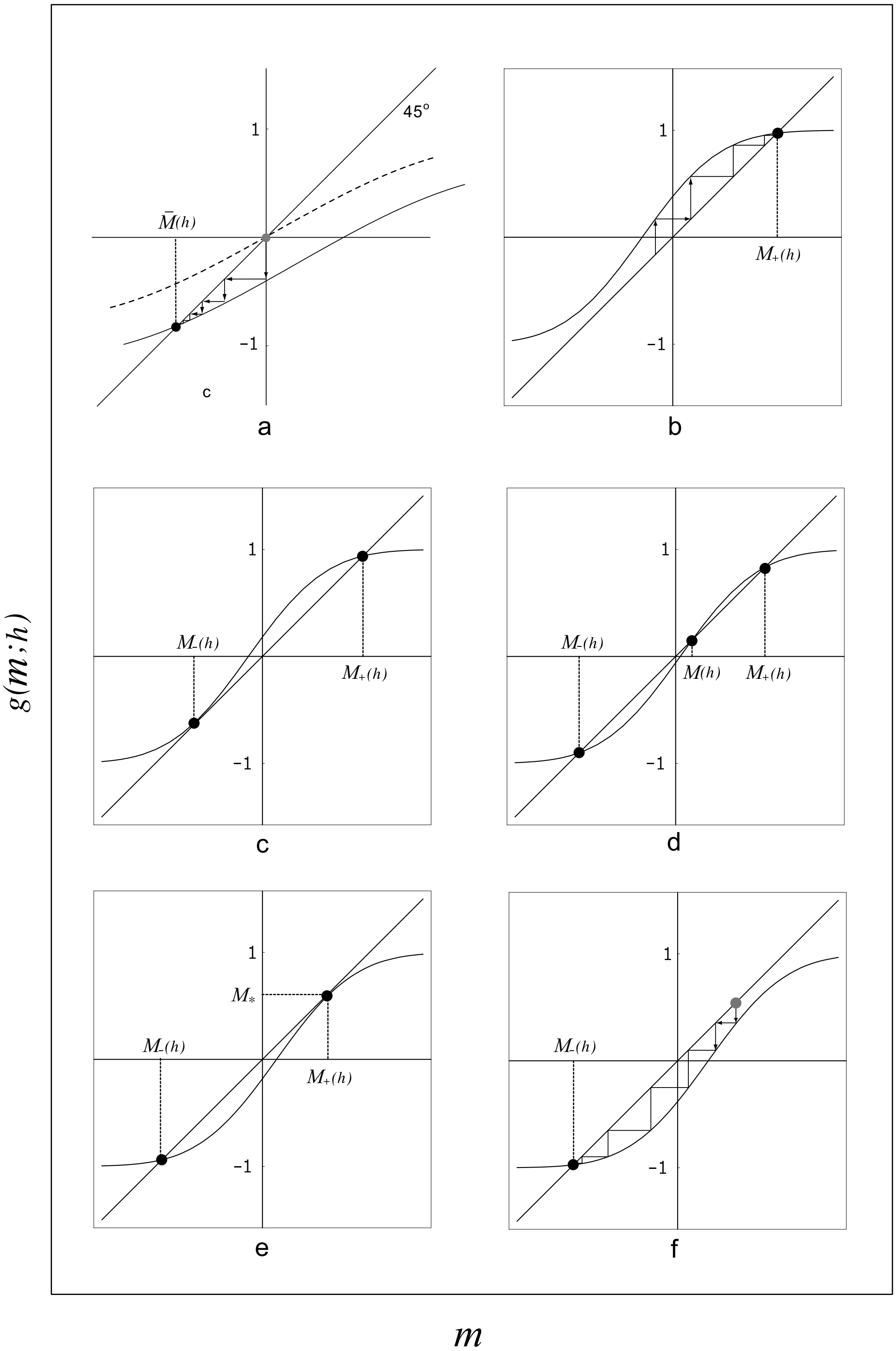

Figure 1: Generic shapes of the graph of $m \mapsto g(m;h) := 2\Phi(Jm-h)-1$ for different values of $\Phi'(0), J$ and h. Case a: $\Phi'(0)^{-1} \geq 2J$. Cases b-f: $\Phi'(0)^{-1} < 2J$. Sub-cases: b: $h < -h_*$; c: $h = -h_*$; d: $-h_* < h < h_*$; e: $h = h_*$; f: $h > h_*$.

In words, h is the expected value of the difference between the price difference $\theta = p(+1) - p(-1)$ and the intrinsic preference $\theta^{(i)}$ of a consumer i.

Recall, according to Remark 1 in Section 2, that the intrinsic preference $\theta^{(i)}$ express the preference of consumer i, when he/she is not affected by the social interaction; i.e., when both establishments are equally demanded, consumer i

1. prefers $E^{(+)}$ to $E^{(-)}$ if the price difference θ is below $\theta^{(i)}$.

2. is indifferent between $E^{(-)}$ and $E^{(+)}$ if $\theta = \theta^{(i)}$.

3. prefers $E^{(-)}$ to $E^{(+)}$ if the price difference θ *exceeds* $\theta^{(i)}$.

In the light of the three assertions above, we will call the quantity $\theta - \theta^{(i)}$ the consumer's i intrinsic excess price difference of the establishments $E^{(+)}$ and $E^{(-)}$. Accordingly, we will call the expected value $h = \mathbb{E}(\theta - \theta^{(i)})$

the excess price difference of establishments (free from social influence)

Now, note that

$$\mathbb{P}(\theta - \theta^{(i)} > 0) = \mathbb{P}(\theta - (\overline{\theta} + \xi^{(i)}) > 0) = \mathbb{P}(\xi^{(i)} < \theta - \overline{\theta}) = \Phi(h) \tag{25}$$

where, due to the symmetry of Φ', we have: $\Phi(h) > 1/2$ ($< 1/2$) if $h > 0$ ($h < 0$). Note also that, due to the continuity of Φ, we have: $\Phi(0) = 1/2$.

Thus, the excess price difference h measures the advantage of one establishment over another. In fact, when the consumers suppose that both establishments are equally demanded (equally attractive from social point of view), the following three relationships holds: i) if $h > 0$ then the majority of consumers will prefer $E^{(-)}$ to $E^{(+)}$ (the exact fraction of consumers that prefer $E^{(-)}$ to $E^{(+)}$ is $\Phi(h) > 1/2$); ii) if $h < 0$ then the majority of consumers will prefer $E^{(+)}$ to $E^{(-)}$ (the exact fraction of consumers that prefer $E^{(+)}$ to $E^{(-)}$ is $1 - \Phi(h) > 1/2$); iii) if $h = 0$ then each establishment attracts 50% of the consumers.

6.2 Two regimes of demand behavior

According to Proposition 2, the system (24) ($m_t = 2\Phi(Jm_{t-1} - h) - 1,\ t \geq 1$) has two regimes:

$\Phi'(0)^{-1} \geq 2J$.

This is the regime where the heterogeneity of the consumers' intrinsic preferences $\Phi'(0)^{-1}$ is relatively *high* compared with social influence on decisions J. We shall call this regime

the high heterogeneity regime ($\Phi'(0)^{-1} \geq 2J$)

In this regime, there is only one stable difference of demand fractions $\bar{M}(h) \in [-1, 1]$ which is odd, decreasing and continuous in the excess price difference $h \in \mathbb{R}$. The generic shape of the function $h \mapsto \bar{M}(h)$ is given in Figure 2.

$\Phi'(0)^{-1} < 2J$.

This is the regime, where the heterogeneity of the consumers' intrinsic preferences $\Phi'(0)^{-1}$ is relatively *low* compared with social influence on decisions J. We shall call this regime

the low heterogeneity regime ($\Phi'(0)^{-1} < 2J$)

In this regime, the stable difference of demand fractions can acquire either one or two values,[3] depending on the value of h. Figure 3 shows schematically this dependence. $M_+(h)$ and $M_-(h)$ show the possible values of the stable difference of demand fractions as functions of the parameter h, In particular, Figure 3 makes clear the following properties that will be important for our socioeconomic interpretation: the functions $M_+(h)$ and $M_-(h)$ are not defined for every h, and they co-exist for $h \in [-h_*, h_*]$; moreover, M_+ is always positive, while M_- is always negative.

The different behaviors of the stable difference of demand fractions in our model and their dependence on the parameters $\Phi'(0)^{-1}$, J and h allow us to link the occurrence of certain market phenomena with the degree of heterogeneity of consumer's intrinsic preferences ($\Phi'(0)^{-1}$), the social influence on consumers'decision (J) and the excess price difference of establishments (h). We shall do this in the following subsections.

6.3 Demand bubble

We stress that in the low heterogeneity regime ($\Phi'(0)^{-1} < 2J$), the stable demand may behave like a stable excess demand of an over-valued asset in a stock market.

Look at the branch of the graph of $M_+(h)$ that lies to the right of the vertical axis in Figure 3. This branch shows that it is possible that the inequalities $M_+(h) > 0$ and $h > 0$ occur simultaneously. Note that the inequality $h > 0$ means that if the consumers were not affected by social interactions, i.e., when they suppose that both establishments are equally demanded, then the majority of them would choose the establishment $E^{(-)}$. On the other hand, the inequality $M_+(h) > 0$ means that the market is in the stable state in which the establishment $E^{(+)}$ is more demanded than $E^{(-)}$.

We note that the demand bubble is impossible in the high heterogeneity regime ($\Phi'(0)^{-1} \geq 2J$). Figure 2 shows this fact clearly: the stable difference of demand fractions $\bar{M}(h)$ is negative when $h > 0$, and $\bar{M}(h)$ is positive when $h < 0$. This means that $E^{(-)}$

[3]Recall that Proposition 2 identifies stable and unstable equilibria of the dynamical system $m_{t+1} = 2\Phi(Jm_{t-1} - h) - 1$. However, only stable equilibria will be viewed as possible values for the stable difference of demand fractions in our model. This is because our model has a stochastic component that would not allow the difference of demand fractions to remain at an unstable equilibrium.

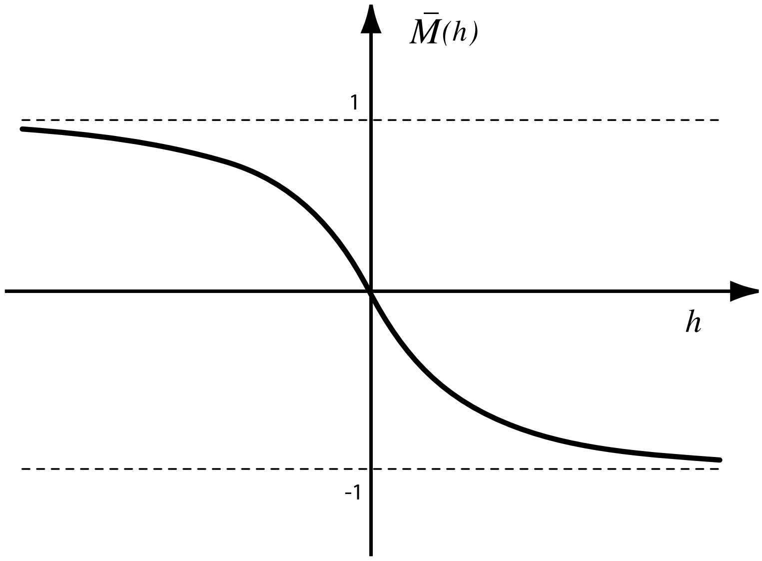

Figure 2: Generic shape of the function $\bar{M}(h)$ that represents the dependence of the globally stable equilibrium of the dynamical system $m_{t+1} = 2\Phi(Jm_t - h) - 1$ on h in the case $\Phi'(0)^{-1} \geq 2J$.

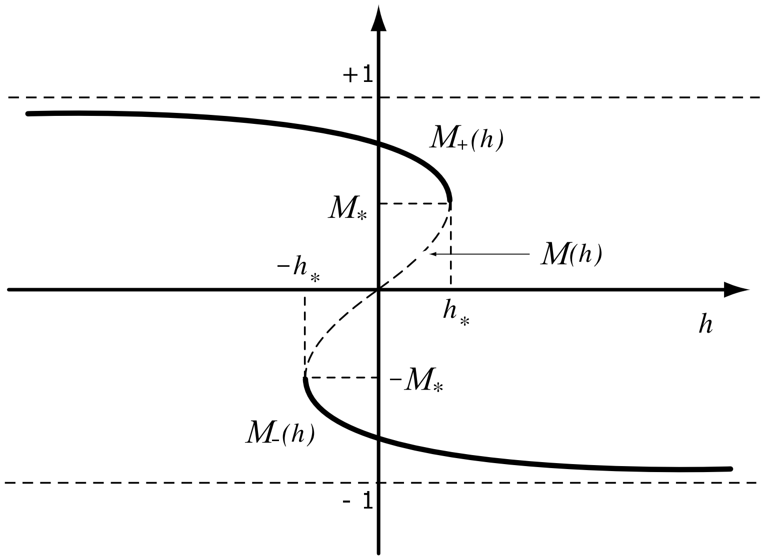

Figure 3: Generic shapes of the function $M_+(h)$, $M(h)$ and $M_-(h)$ that represent the dependence on h of the values of the equilibria of the dynamical system $m_{t+1} = 2\Phi(Jm_t - h) - 1$ in the case $\Phi'(0)^{-1} < 2J$.

is more demanded than $E^{(+)}$ whenever the excess price difference h is positive, and vice versa.

6.4 Working either over or under capacity

When two similar establishments dominate a market, usually each is ready to absorb approximately 50% of the total amount of all virtual consumers of this market (effective consumers of both establishments + queue at the entrance of one of them). This is natural: each manager expects that approximately 50% of the virtual consumers will visit his establishment and the rest will visit the other one. Nevertheless, in real life, it may happen that one of the similar, competing establishments is almost empty, while the other one is full and there is always a queue at its door, even when both charge similar prices. We shall use our model to explain this apparent contradiction.

When both establishments are similar (in an extreme case, identical), it is natural to suppose that all consumers are intrinsically indifferent to both and thus $\theta^{(i)} \simeq 0$, $i \in C$, implying that $\Phi'(0)^{-1} < 2J$. In what follows, we shall show that in the low heterogeneity regime ($\Phi'(0)^{-1} < 2J$), one of the establishments will always work under capacity and the other one over capacity, whenever the capacity of each establishment corresponds to 50% of the total amount of all virtual consumers.

Let us define

$$N^{(+)}(h) := (1 + M_+(h))/2, \text{ and } N^{(-)}(h) := 1 - N^{(+)}(h) \tag{26}$$

where $h \to N^{(+)}(h)$ and $h \to N^{(-)}(h)$ are defined in the domain of the stable difference of demand fractions $h \to M_+(h)$.

Note that $N^{(+)}(h)$ (resp.,$N^{(-)}(h)$) is the fraction of the consumers that visit $E^{(+)}$ (resp.,$E^{(-)}$), when the difference of demand fractions has stabilized at $M_+(h)$. The shapes of the functions $h \to N^{(+)}(h)$ and $h \to N^{(-)}(h)$ are illustrated in Figure 4 and are determined by the shape of function $h \to M_+(h)$, illustrated in Figure 3.

Figure 4 illustrates clearly the fact that there is no excess price difference h for which the stable demand fractions of both establishments would be close to 50% of all virtual consumers (the total amount of N consumers). If both establishments have capacities close to 50% of all virtual consumers, i.e., if the relative capacities $K^{(+)}$ and $K^{(-)}$ of the establishments $E^{(+)}$ and $E^{(-)}$ (absolute capacities divided by the total demand of both) lie between $1/2 - M_*/2$ and $1/2 + M_*/2$ as illustrated in Figure 4, then either $E^{(+)}$ works over capacity and $E^{(-)}$ works under capacity, or vice versa.

We stress that the phenomenon just described does not occur in the high heterogeneity regime ($\Phi'(0)^{-1} \geq 2J$). In the high heterogeneity regime there is a unique globally stable difference of demand fraction $\bar{M}(h)$ which is odd, decreasing and continuous as a function of the excess price difference h. In particular, if the capacities of both establishments correspond to 50% of all demanding consumers, both establishments would work at capacity, whenever the excess price difference vanishes, i.e., $\bar{M}(h) = 0$ whenever $h = 0$.

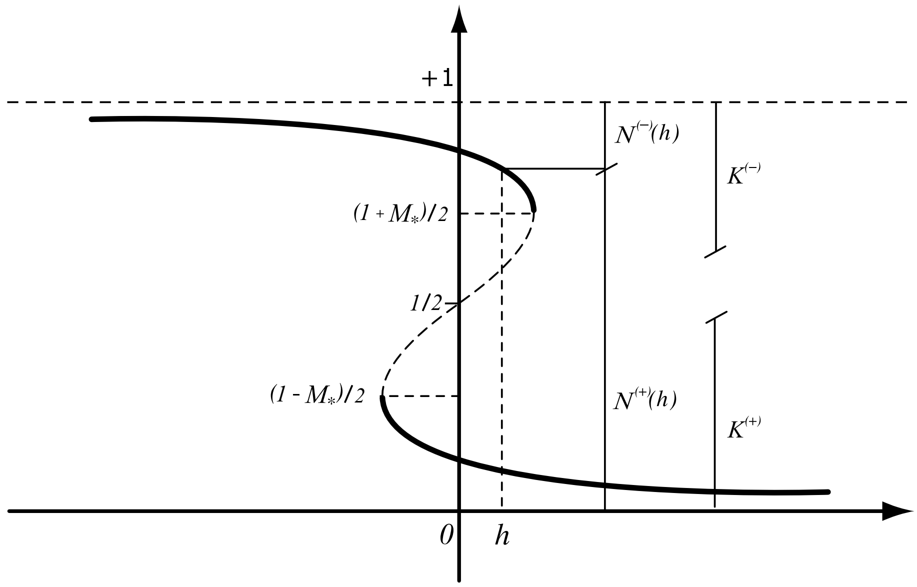

Figure 4: Graphs of $h \mapsto N_+^{(+)}(h)$ and $h \mapsto N_+^{(-)}(h)$.

6.5 Discontinuity of the stable demands and phase change

In this section we shall show that the stable difference of demand fractions is discontinuous in the excess price difference ($h = \theta - \bar{\theta} = p(+1) - p(-1) - \bar{\theta}$) and consequently, that the stable demand fraction of each establishment is a discontinuous function of its price when the low heterogeneity regime prevails ($\Phi(0)^{-1} < 2J$).

We shall also show that the location of the discontinuity point of the stable difference of demand fractions depends on the initial difference of demand fractions at which the system $m_{t+1} = 2\Phi(Jm_t - h) - 1$ is at the initial time $t = 0$. We shall also use this discontinuity to explain why some establishments do not raises their prices even when excess demand persists.

Let us introduce:

$$G(m_0, h) := \lim_{t\to\infty} g_t(m_0, h)$$

where $g_t(m_0, h)$ is the t-th iteration of $m \mapsto 2\Phi(Jm - h) - 1$ starting from m_0.

When $\Phi'(0)^{-1} < 2J$, then the implications (27), (28) and (29) follow from Proposition 2:

$$m_0 > M_* \;\Rightarrow\; G(m_0; h) = \begin{cases} M_+(h) & \text{if} \quad h \le h_* \\ M_-(h) & \text{if} \quad h > h_* \end{cases} \tag{27}$$

(see Figure 5);

$$m_0 < -M_* \;\Rightarrow\; G(m_0; h) = \begin{cases} M_+(h) & \text{if} \quad h < -h_* \\ M_-(h) & \text{if} \quad h \ge -h_* \end{cases} \tag{28}$$

(see Figure 6);

$$-M_* \le m_0 \le M_* \quad \Rightarrow \quad G(m_0; h) = \begin{cases} M_+(h) & \text{if} \quad h < M^{-1}(m_0) \\ m_0 & \text{if} \quad h = M^{-1}(m_0) \\ M_-(h) & \text{if} \quad h > M^{-1}(m_0) \end{cases} \tag{29}$$

(see Figure 7).

Discontinuity. Note that in all three cases $m_0 < -M_*$, $m_0 > -M_*$ and $-M_* \le m_0 \le M_*$, the function $h \mapsto G(m_0; h)$ is discontinuous. In this sense, we say that the stable difference of demand fractions is discontinuous in the excess price difference h. The consequence of the above discontinuity is given in the paragraph below.

Assume without loss of generality that the initial difference of demand fraction is at the stable equilibrium $m_0 = M_+(h)$, with $h < h_*$. Note that (27) shows that a change of the excess price difference from h to $h + \Delta h$, where $h < h + \Delta h < h_*$, leads to a new equilibrium at $M_+(h + \Delta h)$ after a certain relaxation time. When, however, the new excess price difference $h + \Delta h$ exceeds the threshold h_*, the difference of demand fractions m_t is attracted from $M_+(h)$ to the only remaining negative attractor $M_-(h + \Delta h)$, in accordance with (27).

Phase change. Note that, in order to revert the equilibrium from $M_-(h+\Delta h)$ to $M_+(h)$, it is not sufficient to eliminate the increment Δh. Due to the fact that $M_-(h+\Delta) < -M_*$ and the model symmetry, expressed in (28), a change of the excess price difference from $h + \Delta h$ to h, with $-h_* < h < h_* < h + \Delta h$, would lead to a new equilibrium at $M_-(h)$ (and not at $M_+(h)$). To drive the system to the equilibrium at $M_+(h)$ again, one needs first to decrease the excess price difference from $h + \Delta h$ to some h' with $h' < -h_*$ and then bring it back to h. That means that an uppercrossing of the discontinuity point h_* causes not only a discontinuous jump in the current locally stable difference of demand fractions, but it causes also a phase change of the stable difference of demand fractions as a function of h.

Persistence of excess demand in real life. The discontinuity and the phase change mentioned above are consistent with Becker's (1991) argument explaining why restaurants sometimes do not raise prices even when a certain excess demand persists.

In fact, when the demand is a decreasing and discontinuous function of price, the demand size that maximizes the profit may lie above the establishment's capacity which would cause an unavoidable excess demand (queue). When the over demanded establishment makes a mistake and increases its price level above the discontinuity point, it may not only eliminate the excess demand but also reduce effective sales below the profit maximizing sales level. Once the establishments loses its consumers, the old level of effective sales can not be restored by a simple reduction of the prices by the same amount, by which they have been increased. The way back passes through a substantial reduction of the establishments' prices and/or a massive investment in advertising (i.e., a change

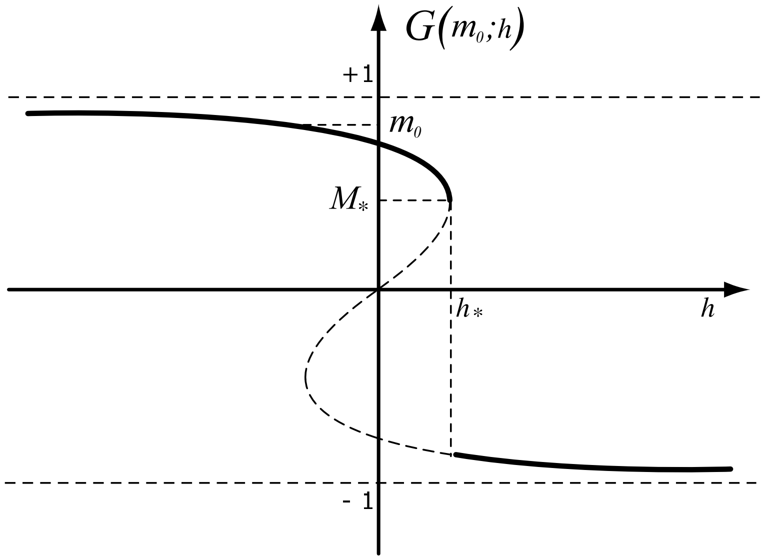

Figure 5: Graph of $h \mapsto G(m_0; h) := lim_{t\to\infty} g_t(m_0; h)$ for $m_0 \in (M_*, 1)$.

of the mean intrinsic preference $\bar{\theta}$) that could bring the excess price difference h below the new discontinuity point $-h_*$. To see this, consider (27) and (28) and recall that $M_-(h) < -M_*$, $\forall h > -h_*$ and $M_+(h) > M_*$, $\forall h < h_*$; and that $h = p(+1) - p(-1) - \bar{\theta}$.

7 Closing remarks

We end our discussion extending the explanation proposed by Gary S. Becker for the original problem that motivated the construction of our model. This problem was proposed by Gary S. Becker (1991) in the following way:

"A popular seafood restaurant in Palo Alto, California, does not take reservations, and every day it has long queues for tables during prime hours. Almost directly across the street is another seafood restaurant with comparable food, slightly higher prices, and similar service and other amenities. Yet this restaurant has many empty seats most of the time.

Why doesn't the popular restaurant raise prices, which would reduce the queue for seats but expand profits ?"

In Section 6.5 we gave our explanation to this question. Here we would like to stress the fact that our explanation to this specific question uses the fact the *two restaurant are similar in their service and amenities, and that they are placed close to one another.*

In fact, when the establishments are similar in their services and amenities and when they are placed close to one another, it is reasonable to assume that all consumers are more or less indifferent between the establishments, when both are equally demanded. In terms of our model this means that the consumers' intrinsic preferences are close to zero ($\theta^{(i)} = \overline{\theta} + \xi^{(i)} \simeq 0$, $i \in C$) and therefore that the dispersion of the consumers

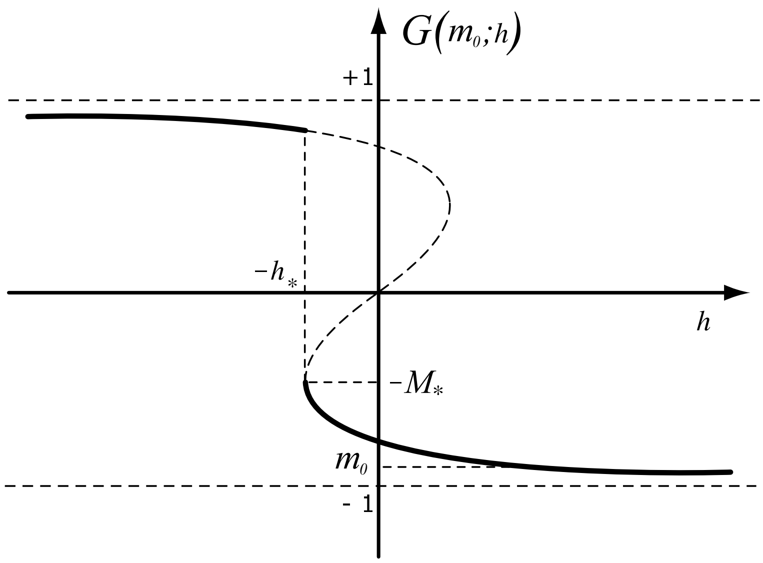

Figure 6: Graph of $h \mapsto G(m_0; h) := lim_{t\to\infty} g_t(m_0; h)$ for $m_0 \in (-1, -M_*)$.

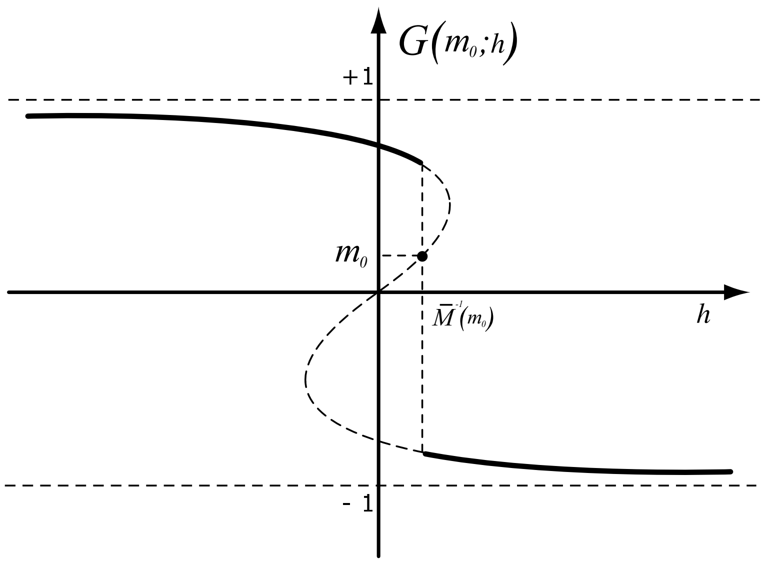

Figure 7: Graph of $h \mapsto G(m_0; h) := lim_{t\to\infty} g_t(m_0; h)$ for $m_0 \in [-M_*, M_*]$.

intrinsic preferences, $\theta^{(i)}$, $i \in C$, is relatively low. Hence, in the case of similar (in an extreme, case identical) restaurants, it is natural to assume that the low heterogeneity regime ($\Phi'(0)^{-1} < 2J$) prevails.

Now, as explained in Section 6.4, under low heterogeneity of consumers' intrinsic preferences, i.e, $\Phi'(0)^{-1} < 2J$, one establishment may work under capacity, while the other works over its capacity; even when both establishments charge the same prices. That the over demanded restaurant does not raise price is due to the profit maximization behavior in the presence of the discontinuity of demand, as explained in Section 6.5. This explanation is consistent with Gary S. Becker's analyzes of demand behavior of two similar restaurants placed close to one an other.

In essence, the novelty of our explanation is that the similarity of the restaurants, i.e., the homogeneity of the consumers's intrinsic preference for the restaurants, plays a key role in guaranteeing the phenomena just described.

Note that we also extend the analysis and results of the demand behavior of two establishments to the case when the establishments (or consumer choices) are dissimilar, i.e., when they represent two distinct consumption options for the population of consumers population. When the consumers are relatively heterogeneous in their intrinsic preferences for establishments ($\Phi'(0)^{-1} \geq 2J$), the two establishments will acquire the same level of demand ($N/2$), whenever there is no excess price difference, i.e, whenever $h = 0$.

It seems, therefore, that the degree of heterogeneity in consumers' intrinsic preferences plays a crucial role in determining the emergence of one or two phases of the demand fractions of the two establishments.

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